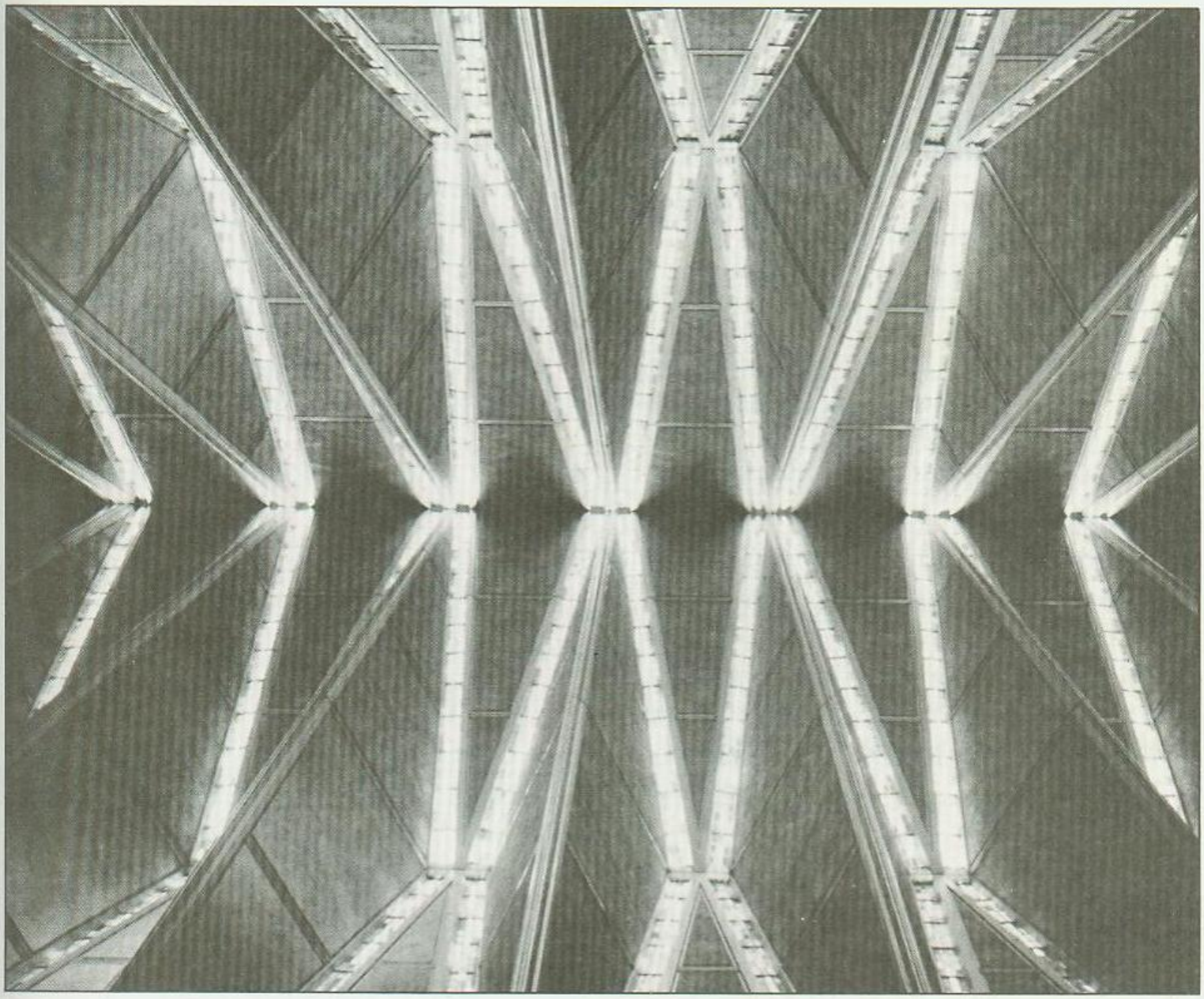
CHAPTER 4

4–1 Steps for Proving a Theorem 130
4–2 Using Addition and Subtraction of Equals Property 138
4–3 Using Supplements and Complements 144
4–4 Using Vertical Angles 150
4–5 Using Exterior Angles 154
4–6 Using Indirect Proof 160

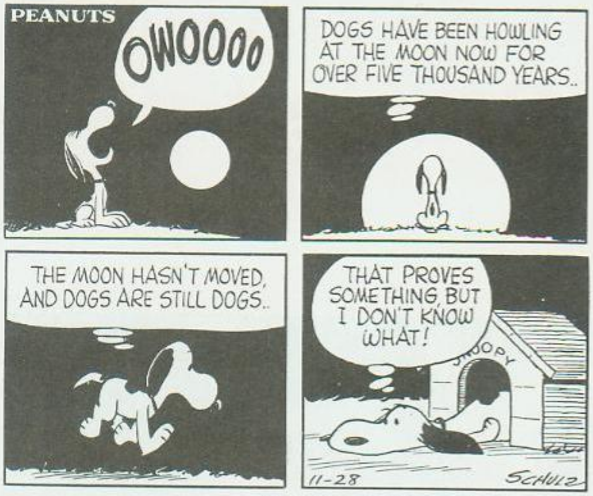
Important Ideas 164 *Review* 165 *Test* 166

Problem Solving Techniques
Make a Table—I 167

Proving Theorems Using Basic Properties



4–1 Steps for Proving a Theorem


PEANUTS
OWOOOO
DOGS HAVE BEEN HOWLING AT THE MOON NOW FOR OVER FIVE THOUSAND YEARS..
THE MOON HASN'T MOVED, AND DOGS ARE STILL DOGS..
THAT PROVES SOMETHING, BUT I DON'T KNOW WHAT!
SNOOPY
11-28
SCHULZ

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In Chapter 2 we said that a theorem is a generalization that can be proved using definitions, postulates, and the logic of deductive reasoning.

In Chapter 3 we used deductive reasoning to write proofs about congruent triangles.

In this lesson we begin using this reasoning process to prove theorems.

The six-step process for proving a theorem will be illustrated in this lesson with several examples. The first theorem is based upon number properties reviewed above.

The reflexive, symmetric, and transitive properties are also true for segment and angle congruence as summarized in this table.

Notice the similarity between the reflexive property and the statement "dogs are dogs."

REVIEW: Some Number Properties

For any numbers, a, b, and c

1. $a = a$ (Reflexive Property)
2. If $a = b$, then $b = a$. (Symmetric Property)
3. If $a = b$ and $b = c$, then $a = c$. (Transitive Property)

Steps For Proving a Theorem

Step 1 If the theorem is not in if-then form, rewrite it in this form.

Step 2 Draw and label a diagram to show the conditions of the theorem.

Step 3 Write the "GIVEN" from the hypothesis ("if" part) of the if-then statement.

Step 4 Write the "PROVE" from the conclusion ("then" part) of the if-then statement.

Step 5 Analyze what is to be proved and devise a plan.

Step 6 Write the proof, giving definitions, postulates, or theorems already proved as reasons.

	Reflexive	Symmetric	Transitive
Angle Congruence	$\angle A \cong \angle A$	If $\angle A \cong \angle B$, then $\angle B \cong \angle A$	If $\angle A \cong \angle B$, and $\angle B \cong \angle C$ then $\angle A \cong \angle C$.
Segment Congruence	$\overline{AB} \cong \overline{AB}$	If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.	If $\overline{AB} \cong \overline{CD}$, and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.

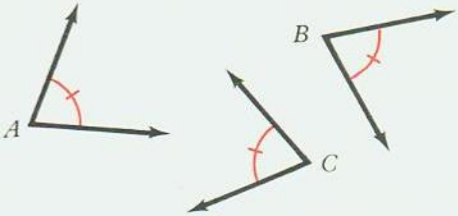
This table actually summarizes six generalizations. One of them is stated and proved below. Notice the six steps for proving a theorem.

When $\angle A$ is congruent to $\angle B$, and $\angle B$ is congruent to $\angle C$, it is also true that $\angle A$ is congruent to $\angle C$.

PROOF

Step 1 If $\angle A$ is congruent to $\angle B$ and $\angle B$ is congruent to $\angle C$, then $\angle A$ is congruent to $\angle C$.

Step 2



Step 3 Given: $\angle A \cong \angle B$
$\angle B \cong \angle C$

Step 4 Prove: $\angle A \cong \angle C$

Step 5 Devise a plan.

If I interpret each of the GIVEN statements as statements about angle measure, then I could use the transitive property of numbers. And I know that congruent angles do have equal measure by definition of congruent angles.

Step 6

Statements	Reasons
1. $\angle A \cong \angle B$	1. Given
2. $m\angle A = m\angle B$	2. Definition of congruent angles
3. $\angle B \cong \angle C$	3. Given
4. $m\angle B = m\angle C$	4. Why?
5. $m\angle A = m\angle C$	5. Transitive property of numbers
6. $\angle A \cong \angle C$	6. Why?

A similar proof could be given for the other five statements summarized by the chart. We combine all these statements and state the following theorem.

Theorem 4–1 The reflexive, symmetric, and transitive properties hold for angle and segment congruence.

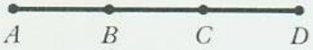
The next two theorems further illustrate the six step process for proving a theorem.

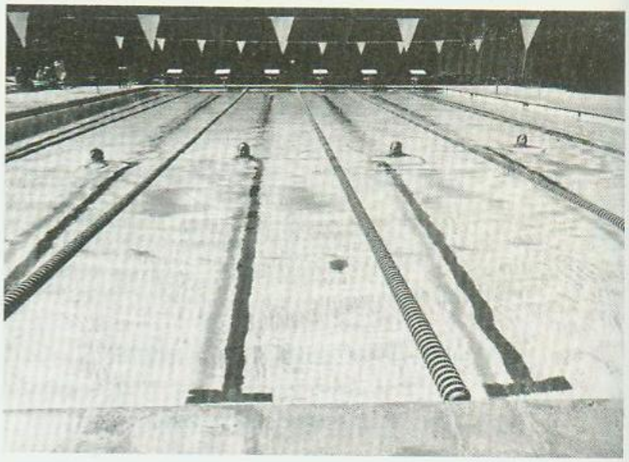
Theorem: If points A, B, C, and D are on a line so that B is the midpoint of $\overline{AC}$ and C is the midpoint of $\overline{BD}$, then $\overline{AB} \cong \overline{CD}$.

PROOF

Step 1 Already done by the statement of the theorem.

Step 2





Step 3 **Given:** B is the midpoint of $\overline{AC}$
C is the midpoint of $\overline{BD}$

Step 4 **Prove:** $\overline{AB} \cong \overline{CD}$

Step 5 Devise a plan.

I shall interpret each of the given statements as a statement about segment congruence. Then I shall use the fact that segment congruence satisfies the transitive property.

Step 6

Statements	Reasons
1. B is the midpoint of $\overline{AC}$	1. Given
2. C is the midpoint of $\overline{BD}$	2. Why?
3. $\overline{AB} \cong \overline{BC}$	3. Definition of midpoint
4. $\overline{BC} \cong \overline{CD}$	4. Why?
5. $\overline{AB} \cong \overline{CD}$	5. Transitive property (Theorem 4-1)

The reason in Step 5 is a theorem that has already been proved. We have been using the postulates, definitions, and the given as reasons in proofs up to this point. The proof above shows that theorems already proved are also an important part of the deductive reasoning process.

When you are devising a plan for a proof, review definitions, postulates, and theorems that have already been proved.

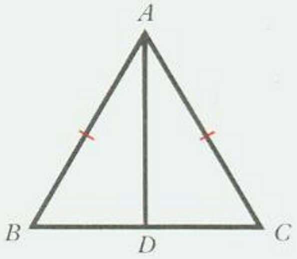
Theorem 4-2 In an isosceles triangle the segment from the vertex angle to the midpoint of the opposite side forms a pair of congruent triangles.

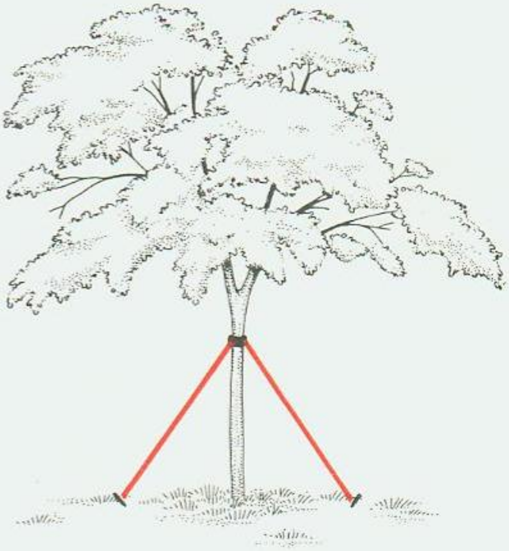
The statement of this theorem is complex enough to illustrate the importance of steps 1 and 2 in this six step process.

PROOF

Step 1 If $\triangle ABC$ is an isosceles triangle with $\overline{AB} \cong \overline{AC}$ and if D is the midpoint of $\overline{BC}$, then $\triangle ABD \cong \triangle ACD$.

Step 2





Step 3 **Given:** $\overline{AB} \cong \overline{AC}$
D is the midpoint of $\overline{BC}$.

Step 4 **Prove:** $\triangle ABD \cong \triangle ACD$

Step 5 Devise a plan.

I shall use the given information, the definition of midpoint, and the reflexive property of segment congruence together with the SSS Congruence Postulate.

Step 6

Statements	Reasons
1. $\overline{AB} \cong \overline{AC}$	1. Given
2. D is midpoint of $\overline{BC}$	2. Given
3. $\overline{BD} \cong \overline{CD}$	3. Definition of midpoint
4. $\overline{AD} \cong \overline{AD}$	4. Reflexive property of segment congruence (Theorem 4-1)
5. $\triangle ABD \cong \triangle ACD$	5. SSS Congruence Postulate

EXERCISES

A.

In exercises 1–3 a theorem is given in if-then form. Draw a picture and state the "Given" and "Prove" using the picture and its labeling. *Do not prove any of the theorems.*

1. Theorem. If $\triangle ABC$ is an isosceles triangle, then $\triangle ABC$ has a pair of congruent angles.
2. Theorem. If points X, Y, Z are midpoints of the sides of $\triangle ABC$, then segments $\overline{XY}$, $\overline{XZ}$, and $\overline{YZ}$ divide $\triangle ABC$ into four congruent triangles.
3. Theorem. If X and Y are the midpoints of two sides of a triangle, then XY is equal to one half the length of the third side.

In exercises 4–7, restate the theorem in if-then form, draw a picture, and state the "Given" and "Prove." *Do not attempt to prove the theorems.*

4. An equilateral triangle is an isosceles triangle.
5. Two intersecting lines form two pairs of congruent angles.
6. Two intersecting lines that are not perpendicular form a pair of obtuse angles.
7. An angle bisector of a vertex angle in an equilateral triangle is a perpendicular bisector of a side.

Name the property of segment or angle congruence (reflexive, symmetric, or transitive) illustrated by each statement below.

8. If two segments are congruent, the congruence statement may be written with either segment first.
9. Any segment is congruent to itself.
10. If a first angle is congruent to a second angle and the second angle is congruent to a third angle, then the first angle is congruent to the third angle.

In exercises 11–14 use the "Given" and "Prove" to draw the picture. Then write a general theorem. *Do not prove any of these theorems.*

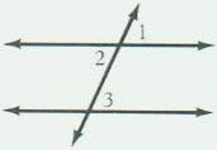
11. **Given:** $\triangle ABC$ is an isosceles triangle with $\overline{AB} \cong \overline{AC}$.
    Prove: $\angle B \cong \angle C$
12. **Given:** $\triangle ABC$ with $\angle B \cong \angle C$
    Prove: $\triangle ABC$ is an isosceles triangle.

13. **Given:** $\triangle ABC$ is an equilateral triangle.
Prove: $\angle A \cong \angle B \cong \angle C$

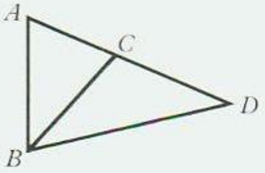
14. **Given:** $\triangle ABC$ with $\angle A \cong \angle B \cong \angle C$
Prove: $\triangle ABC$ is an equilateral triangle.

B.

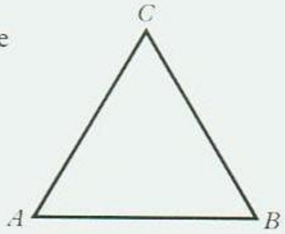
15. If $\angle 1 \cong \angle 2$ and $\angle 2 \cong \angle 3$, why is $\angle 1 \cong \angle 3$?



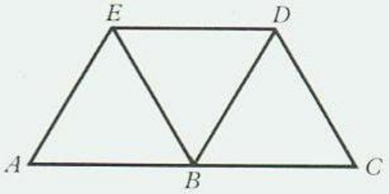
16. If $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{CD}$, why is $\overline{AB} \cong \overline{CD}$?



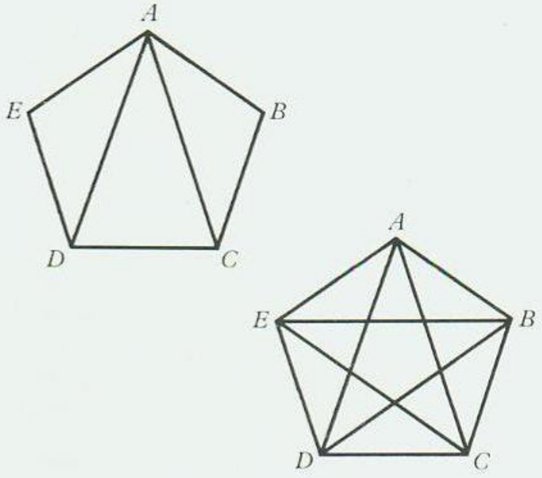
17. If $AB = BC$ and $BC = AC$, why do we know that $\triangle ABC$ is equilateral?



18. If $\overline{AB} \cong \overline{ED}$ and $\overline{ED} \cong \overline{BC}$, how do you know that B is the midpoint of $\overline{AC}$?



19. In a regular pentagon two diagonals from the same vertex are congruent. For example, $\overline{AC} \cong \overline{AD}$ in the first figure. Use this fact with a transitive property to explain why $\overline{AC}$ and $\overline{BE}$ are congruent in the second figure.

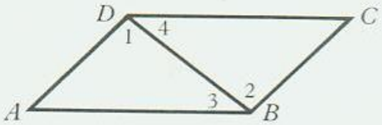


20. A plumber is measuring the length of $\overline{AB}$ in order to cut a piece of pipe of the correct length. The pipe that is cut is called $\overline{CD}$. How does the transitive property for congruence of segments show that the cut pipe $\overline{CD}$ will fit the space $\overline{AB}$? (*Hint:* The tape measure is an important part of the reasoning.)

C.

For exercises 21–22, write a complete two-column proof. Each proof requires the use of one of the congruence properties for segments or angles specified in Theorem 4–1.

21.

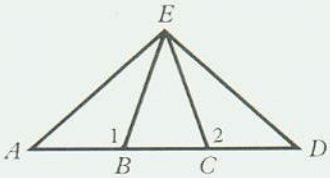


Given: $\angle 1 \cong \angle 2$

$\angle 3 \cong \angle 4$

Prove: $\triangle ABD \cong \triangle CDB$

22.



Given: $\overline{BE} \cong \overline{CE}$

$\angle 1 \cong \angle 2$

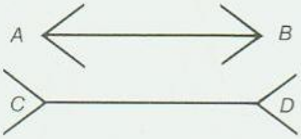
B is the midpoint of $\overline{AC}$

C is the midpoint of $\overline{BD}$

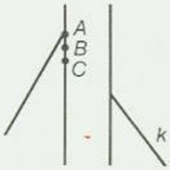
Prove: $\triangle ABE \cong \triangle DCE$

Activity

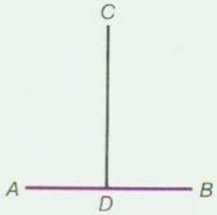
These Optical Illusions show why we should rely more on logical reasoning and less on visual information. Answer the questions. Then check by measuring.



a. Is it further from A to B or C to D?



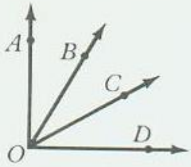
b. Will line k, when extended, meet point A, point B, point C, or none of these points?



c. Which segment is longer, $\overline{AB}$ or $\overline{CD}$?

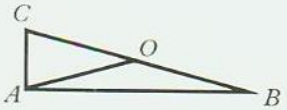
Create an optical illusion of your own.

23.



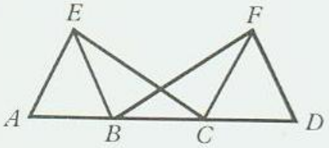
Given: $\overrightarrow{OB}$ bisects $\angle AOC$
$\overrightarrow{OC}$ bisects $\angle BOD$
Prove: $\angle AOB \cong \angle COD$

24.



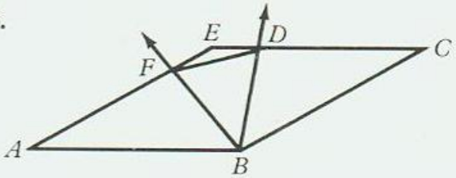
Given: O is midpoint of $\overline{BC}$
$\triangle AOB$ is isosceles with $\overline{OA} \cong \overline{OB}$
Prove: $\triangle AOC$ is isosceles

25.



Given: $\triangle ACE \cong \triangle DBF$
B is midpoint of $\overline{AC}$
C is midpoint of $\overline{BD}$
Prove: $\triangle ABE \cong \triangle DCF$

26.

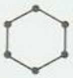


Given: $ABCE$ with $\overline{AB} \cong \overline{BC}$
$\triangle BDF$ is isosceles with $\overline{BF} \cong \overline{BD}$
$\overrightarrow{BF}$ bisects $\angle ABD$ and
$\overrightarrow{BD}$ bisects $\angle CBF$
Prove: $\triangle ABF \cong \triangle CBD$

27. Prove that congruence of triangles satisfies the transitive property.
28. Prove that all right angles are congruent.

PROBLEM SOLVING

1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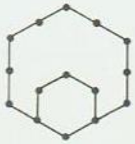
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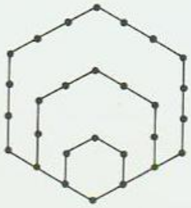
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• • •

The first four hexagonal numbers are shown above.

a. Give the next 2 hexagonal numbers. Show the dot patterns.

b. Does the formula $n(2n - 1)$ give the n^{th} hexagonal number?

4–2 Using Addition and Subtraction of Equals Property

The properties of numbers on the right review some algebra.

These properties are used in proofs that involve lengths of segments and measures of angles. The two theorems in this lesson illustrate the use of these properties.

In proofs later in the book you will often use these theorems rather than the number properties themselves.

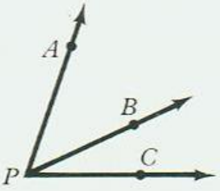
REVIEW: Some Number Properties

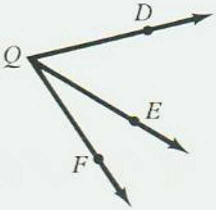
Addition of Equals: If $a = b$ and $c = d$, then $a + c = b + d$.
Subtraction of Equals: If $a = b$ and $c = d$, $a > c$, then $a - c = b - d$.
Multiplication of Equals: If $a = b$ and $c = d$, then $a \cdot c = b \cdot d$.
Substitution Principle: If $a = b$, then a may be replaced by b in any equation or inequality.

Theorem 4–3 **Addition of Equal Angles.** If $m\angle APB = m\angle DQE$, $m\angle BPC = m\angle EQF$, $\overrightarrow{PB}$ is between $\overrightarrow{PA}$ and $\overrightarrow{PC}$, and $\overrightarrow{QE}$ is between $\overrightarrow{QD}$ and $\overrightarrow{QF}$, then $m\angle APC = m\angle DQF$.

PROOF

Given: $m\angle APB = m\angle DQE$
$m\angle BPC = m\angle EQF$
Prove: $m\angle APC = m\angle DQF$



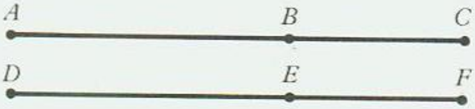


Statements	Reasons
1. $m\angle APB = m\angle DQF$	1. Given
2. $m\angle BPC = m\angle EQF$	2. Given
3. $m\angle APB + m\angle BPC = m\angle DQE + m\angle EQF$	3. Addition of Equals Property
4. $m\angle APB + m\angle BPC = m\angle APC$	4. Definition of between for rays
5. $m\angle DQE + m\angle EQF = m\angle DQF$	5. Why?
6. $m\angle APC = m\angle DQF$	6. Substitution Principle

In general, information telling which rays are between others is not stated in the "Given." It can be taken from the figures. The same is true about points, as illustrated in the next theorem.

Theorem 4–4 **Subtraction of Equal Segments.** If $AC = DF$, $BC = EF$, B is between A and C, and E is between D and F, then $AB = DE$.

Given: $AC = DF$
$BC = EF$
Prove: $AB = DE$



Statements	Reasons
1. $AC = DF$	1. Given
2. $BC = EF$	2. Given
3. $AC = AB + BC$	3. Definition of between for points
4. $DF = DE + EF$	4. Why?
5. $AB + BC = DE + EF$	5. Substitution Principle
6. $AB + BC - BC = DE + EF - EF$	6. Subtraction of Equals Property
7. $AB + BC - BC = AB$	7. Properties of algebra
8. $DE + EF - EF = DE$	8. Why?
9. $AB = DE$	9. Substitution Principle

The next two theorems are stated without proof.

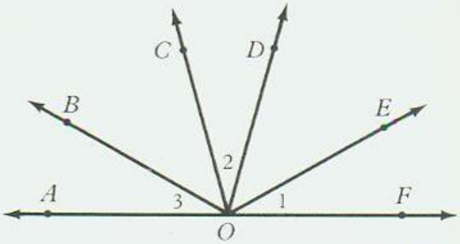
Theorem 4–5 **Addition of Equal Segments.** If $AB = DE$, $BC = EF$, B is between A and C, and E is between D and F, then $AC = DF$.

Theorem 4–6 **Subtraction of Equal Angles.** If $m\angle APC = m\angle DQF$, $m\angle BPC = m\angle EQF$, $\overrightarrow{PB}$ is between $\overrightarrow{PA}$ and $\overrightarrow{PC}$, and $\overrightarrow{QE}$ is between $\overrightarrow{QD}$ and $\overrightarrow{QF}$, then $m\angle APB = m\angle DQE$.

EXERCISES

A.

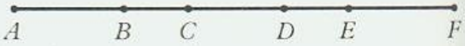
Are the conclusions for exercises 1–6 justified by the transitive property for equality, the Addition of Equal Angles Theorem, or the Subtraction of Equal Angles Theorem?



(Exs. 1–6)

1. If $m\angle 1 = m\angle 2$, and $m\angle 2 = m\angle 3$, then $m\angle 1 = m\angle 3$.
2. If $m\angle COD = m\angle EOF$, then $m\angle COE = m\angle DOF$.
3. If $m\angle BOD = m\angle COE$, then $m\angle BOC = m\angle DOE$.
4. If $m\angle AOC = m\angle DOF$ and $m\angle 1 = m\angle 3$, then $m\angle BOC = m\angle DOE$.
5. If $m\angle AOC = m\angle BOD$ and $m\angle BOD = m\angle DOE$, then $m\angle AOC = m\angle DOE$.
6. If $m\angle AOD = m\angle FOC$ and $m\angle BOD = m\angle EOC$, then $m\angle AOB = m\angle FOE$.

For exercises 7–10, state a property, a theorem, or a combination of them that justifies the given statements.

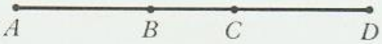


(Exs. 7–10)

7. If $BD = CE$, then $BC = DE$.
8. If $AC = DF$, then $AD = CF$.
9. If $BE = DF$, and $DF = AC$, then $CE = AB$.
10. If $AB = EF$ and $BC = DE$, then $AD = CF$.

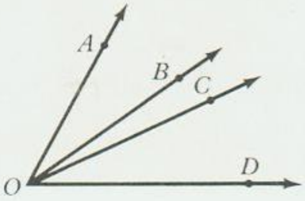
In exercises 11–14 provide all the missing reasons in the proofs.

11. **Given:** $AB = CD$
    Prove: $AC = BD$



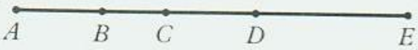
Statements	Reasons
1. $AB = CD$	1. Given
2. $BC = BC$	2. ?
3. $AC = BD$	3. ?

12. **Given:** $m\angle AOC = m\angle BOD$
    Prove: $m\angle AOB = m\angle COD$



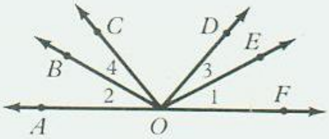
Statements	Reasons
1. $m\angle AOC = m\angle BOD$	1. Given
2. $m\angle BOC = m\angle BOC$	2. ?
3. $m\angle AOB = m\angle COD$	3. ?

13. Given: $AB = CD$
$BD = DE$
Prove: $AC = DE$



Statements	Reasons
1. $AB = CD$	1. Given
2. $BD = DE$	2. Given
3. $BC = BC$	3. ?
4. $AC = BD$	4. ?
5. $AC = DE$	5. ?

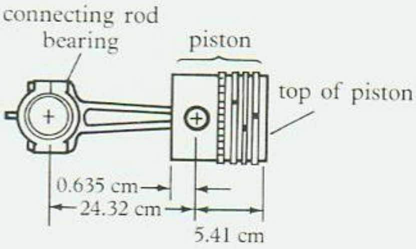
14. Given: $m\angle AOD = m\angle FOC$
$m\angle 3 = m\angle 4$
Prove: $m\angle 1 = m\angle 2$



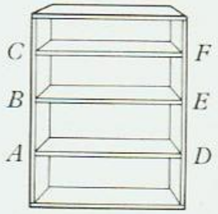
Statements	Reasons
1. $m\angle AOD = m\angle FOC$	1. ?
2. $m\angle COD = m\angle COD$	2. ?
3. $m\angle AOC = m\angle FOD$	3. ?
4. $m\angle 3 = m\angle 4$	4. ?
5. $m\angle 2 = m\angle 1$	5. ?

B.

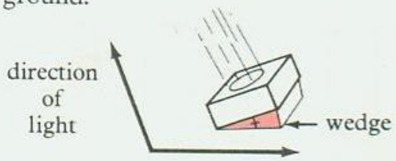
15. Consider the diagram at the right.
    - **a.** What is the distance from the center of the connecting rod bearing to the top of the piston?
    - **b.** What is the length of the piston?
    - **c.** What theorems or postulates support your answers?

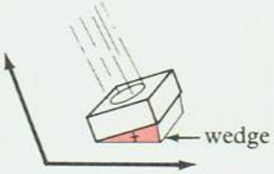


16. Shelf brackets are to be mounted in a book shelf. If they are mounted so that $AB = DE$ and $BC = EF$, what theorem are you using to conclude that the first and third shelves are an equal distance apart at both ends?



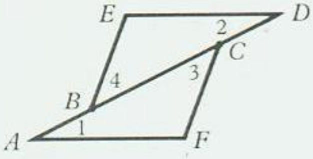
17. Two identical outdoor lights are each enclosed in a box. Two identical wedges are placed under the lights. What theorem assures that the two beams of light form the same angle with the ground?



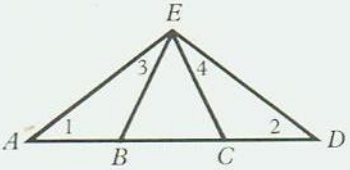


In exercises 18–27 write a complete two-column proof.

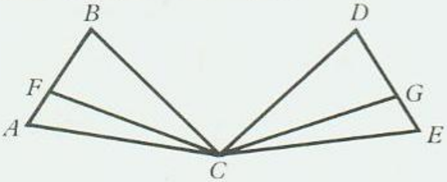
18. Given: $\overline{AB} \cong \overline{CD}$
$\angle 1 \cong \angle 2$
$\angle 3 \cong \angle 4$
Prove: $\triangle ACF \cong \triangle DBE$



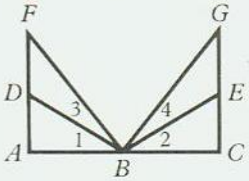
19. Given: $\overline{AE} \cong \overline{DE}$
$\angle 1 \cong \angle 2$
$\angle 3 \cong \angle 4$
Prove: $\overline{AC} \cong \overline{BD}$



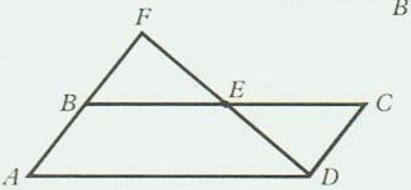
20. Given: $\triangle BFC \cong \triangle DGC$
$\triangle ABC \cong \triangle EDC$
Prove: $\triangle AFC \cong \triangle EGC$



21. Given: B is the midpoint of $\overline{AC}$.
$\angle 1 \cong \angle 2$
$\angle 3 \cong \angle 4$
$\overline{FB} \cong \overline{GB}$
Prove: $\overline{FD} \cong \overline{GE}$



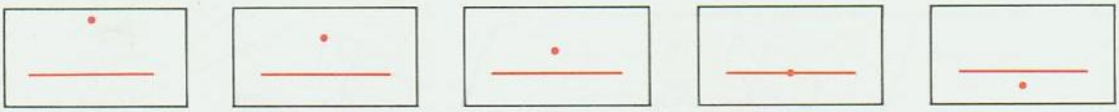
22. Given: B is the midpoint of $\overline{AF}$.
E is the midpoint of $\overline{BC}$.
E is the midpoint of $\overline{DF}$.
$\overline{AB} \cong \overline{CD}$
Prove: $\triangle BEF \cong \triangle CED$



Activity

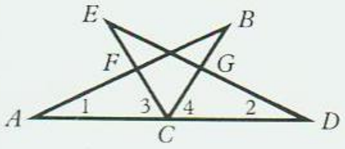
A set of points that uniquely satisfies a given condition is called a **locus**. The locus of points equidistant from the end points of a line segment is the perpendicular bisector of the segment.

You can demonstrate that this is true by watching the figure traced by a "moving point." Make a special set of 20 cards (8 cm × 13 cm), clip them together, and flip through them with your thumb to see the moving point! To see the perpendicular bisector use a series of cards like these.

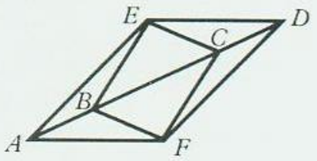


C.

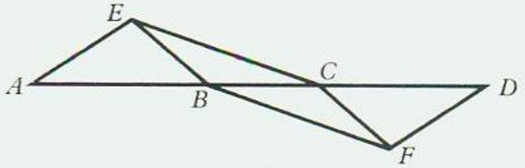
23. Given: C is midpoint of $\overline{AD}$.
$\angle 1 \cong \angle 2$
$\angle 3 \cong \angle 4$
Prove: $\triangle ECG \cong \triangle BCF$



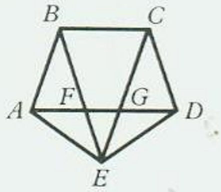
24. Given: $\triangle AEC \cong \triangle DFB$
Prove: $\triangle ABF \cong \triangle DCE$



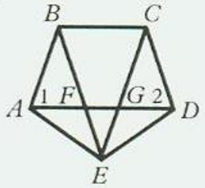
25. Given: $\triangle AEC \cong \triangle DFB$
Prove: $\triangle ABE \cong \triangle DCF$



26. Given: $ABCDE$ is a regular pentagon.
$\overline{FE} \cong \overline{GE}$
Prove: $\triangle ABF \cong \triangle DCG$

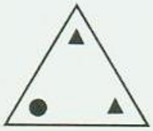


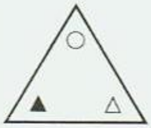
27. Given: $ABCDE$ is a regular pentagon.
$\angle 1 \cong \angle 2$
Prove: $\triangle FGE$ is isosceles

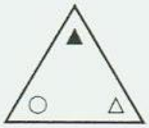


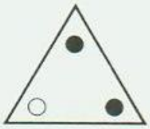
PROBLEM SOLVING

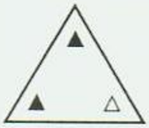
Arrange these nine triangles to form a large triangle so that vertices that touch all have the same symbol.

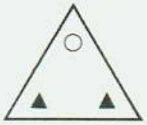


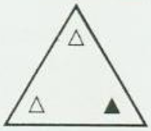


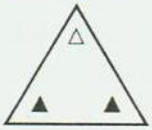


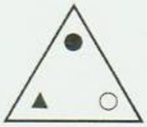






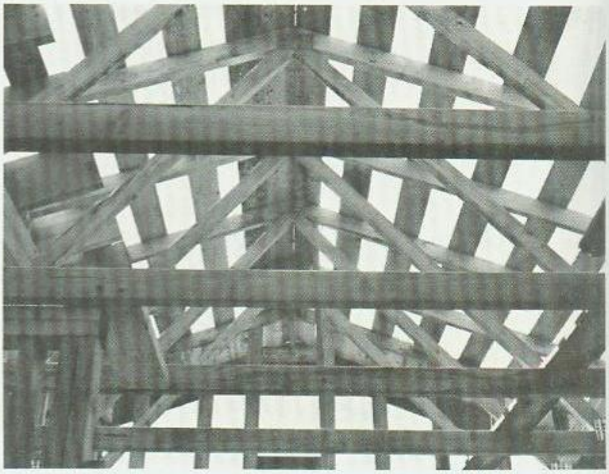


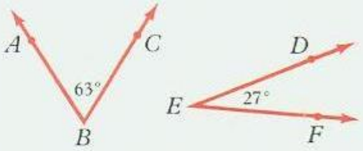




4-3 Proving Theorems—Using Supplements and Complements

Often in the physical world angles occur in pairs whose degree measures have a sum of either 180 or 90. Pairs of this type are studied in this lesson.



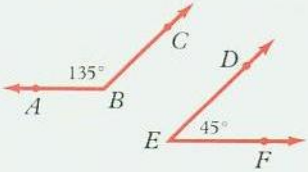


The sum of the measures of $\angle ABC$ and $\angle DEF$ is 90°.

$\angle ABC$ is complementary to $\angle DEF$.
$\angle DEF$ is complementary to $\angle ABC$.

Definition 4-1

Complementary angles are two angles whose measures have a sum of 90°.



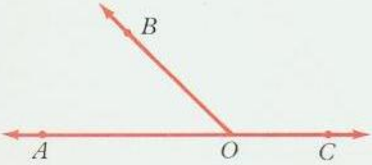
The sum of the measures of $\angle ABC$ and $\angle DEF$ is 180°.

$\angle ABC$ is supplementary to $\angle DEF$.
$\angle DEF$ is supplementary to $\angle ABC$.

Definition 4-2

Supplementary angles are two angles whose measures have a sum of 180°.

Some pairs of angles with a sum of 180° have a common vertex, a common side, and no interior points in common. They are called a linear pair. Find a linear pair of angles in the photograph at the top of the page.



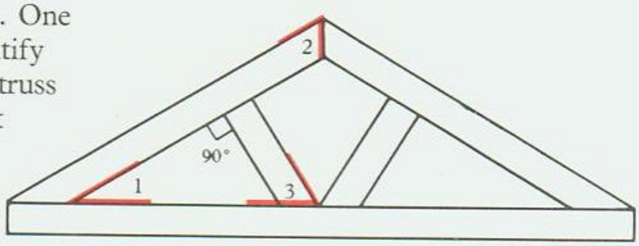
$\angle AOB$ and $\angle BOC$ have a common side, $\overrightarrow{OB}$. The union of the other two sides, $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is a line.

$\angle AOB$ and $\angle BOC$ are a linear pair of angles.

Definition 4-3

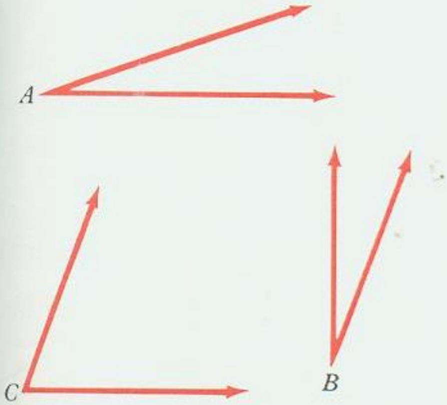
A linear pair of angles is a pair of angles with a common side such that the union of the other two sides is a line.

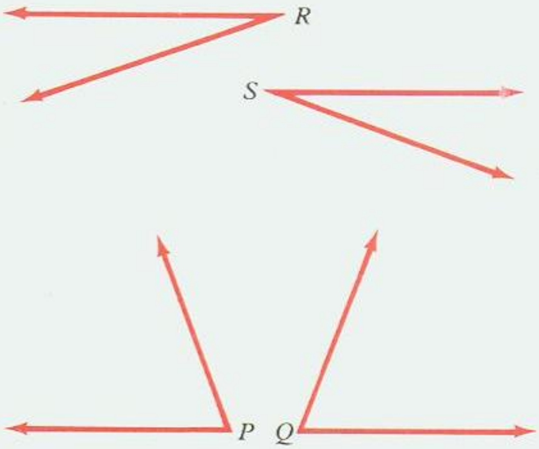
The framework that supports the roof of a house is often assembled separately. This framework is called a roof truss system. One task of a production engineer is to identify all angles of the same size in this roof truss system. Then all boards with congruent angles can be cut at the same time.



The theorem in this lesson provides information that is useful in determining the size of angles.

Use your protractor to answer these questions about complements.





Is $\angle A$ complementary to $\angle C$?
Is $\angle B$ complementary to $\angle C$?
How does $m\angle A$ compare with $m\angle B$?

Is $\angle P$ complementary to $\angle R$?
Is $\angle Q$ complementary to $\angle S$?
How does $m\angle P$ compare with $m\angle Q$?

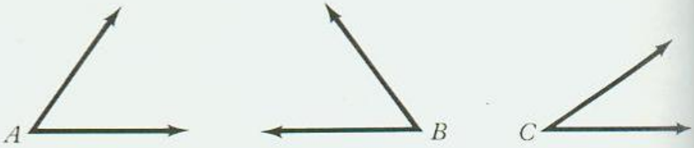
You may have said that $\angle A$ and $\angle B$ are congruent and $\angle P$ and $\angle Q$ are congruent. Theorem 4-7 summarizes the two situations discussed above.

Theorem 4–7 **Congruent Complements Theorem.** Two angles that are complementary to the same angle (or congruent angles) are congruent.

We shall prove the first part of the Congruent Complements Theorem. The second part will be completed as an exercise.

PROOF

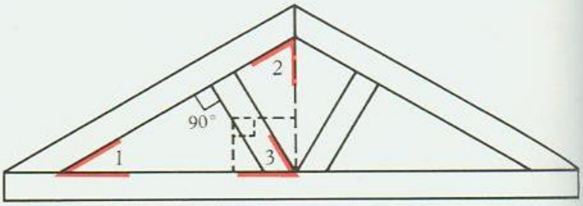
Given: $\angle A$ is complementary to $\angle C$
$\angle B$ is complementary to $\angle C$
Prove: $\angle A \cong \angle B$



Statements	Reasons
1. $\angle A$ is complementary to $\angle C$	1. Given
2. $m\angle A + m\angle C = 90$	2. Definition of complementary angles
3. $\angle B$ is complementary to $\angle C$	3. Why?
4. $m\angle B + m\angle C = 90$	4. Why?
5. $m\angle A + m\angle C = m\angle B + m\angle C$	5. Substitution Principle
6. $m\angle A = m\angle B$	6. Subtraction of Equals Property
7. $\angle A \cong \angle B$	7. Definition of congruent angles

APPLICATION

In a triangle with a 90° angle, the sum of the other two angles is also 90°. Consequently, $\angle 2$ and $\angle 3$ in this roof truss system are both complements of $\angle 1$. By Theorem 4-7 we can conclude that $\angle 2$ and $\angle 3$ have the same measure.



Theorem 4–8 **Congruent Supplements Theorem.** Two angles that are supplementary to the same angle (or to congruent angles) are congruent.

It seems reasonable that two angles that form a linear pair should be supplementary. This fact is accepted as true and called the Linear Pair Postulate.

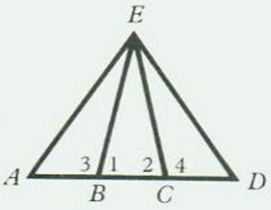
Linear Pair Postulate

If two angles form a linear pair, the angles are supplementary.

Study each step in the proofs below and supply the missing reasons.

Example 1

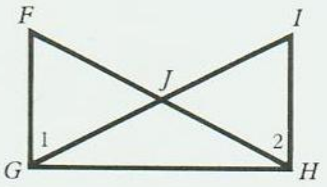
Given: $\angle A \cong \angle D$
$\angle 1 \cong \angle 2$
$\overline{AB} \cong \overline{CD}$
Prove: $\overline{BE} \cong \overline{CE}$



Statements	Reasons
1. $\angle A \cong \angle D$	**1.** Given
2. $\overline{AB} \cong \overline{CD}$	**2.** Why?
3. $\angle 1 \cong \angle 2$	**3.** Why?
4. $\angle 1$ and $\angle 3$ are supplementary	**4.** Linear Pair Postulate
5. $\angle 2$ and $\angle 4$ are supplementary	**5.** Why?
6. $\angle 3 \cong \angle 4$	**6.** Congruent Supplements Theorem
7. $\triangle ABE \cong \triangle DCE$	**7.** Why?
8. $\overline{BE} \cong \overline{CE}$	**8.** CPCTC

Example 2

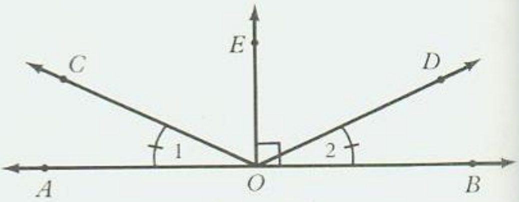
Given: $\angle FGH$ and $\angle IHG$ are right angles.
$\angle 1 \cong \angle 2$
Prove: $\triangle IHG \cong \triangle FGH$



Statements	Reasons
1. $\angle FGH$ and $\angle IHG$ are right angles.	**1.** Given
2. $\angle FGH \cong \angle IHG$	**2.** All right angles are congruent.
3. $m\angle FGH = m\angle IHG = 90$	**3.** Why?
4. $\angle 1 \cong \angle 2$	**4.** Given
5. $\angle 1$ and $\angle IGH$ are complementary.	**5.** Definition of complements
6. $\angle 2$ and $\angle FHG$ are complementary.	**6.** Why?
7. $\angle IGH \cong \angle FHG$	**7.** Congruent Complements Theorem
8. $\overline{GH} \cong \overline{HG}$	**8.** Reflexive property (Segments)
9. $\triangle IHG \cong \triangle FGH$	**9.** Why?

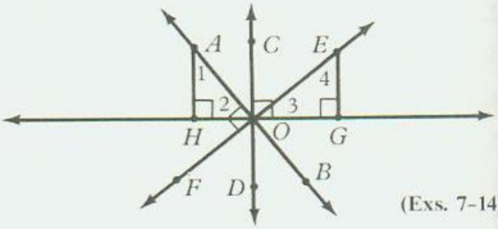
EXERCISES

A.



$\overleftrightarrow{AB} \perp \overrightarrow{OE}$
$\angle 1 \cong \angle 2$
(Exs. 1-6)

1. Name a supplement of $\angle 1$.
2. Name a supplement of $\angle COB$.
3. Name a complement of $\angle COE$.
4. Name a complement of $\angle 2$.
5. Why are $\angle COE$ and $\angle DOE$ congruent?
6. Why are $\angle COB$ and $\angle AOD$ congruent?



$\overleftrightarrow{AB} \perp \overleftrightarrow{EF}$, $\overleftrightarrow{CD} \perp \overleftrightarrow{HG}$
$\overline{AH} \perp \overline{HG}$, $\overline{EG} \perp \overline{HG}$
(Exs. 7-14)

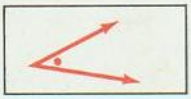
7. Name two angles that are supplements of $\angle COE$.
8. Name two angles that are complements of $\angle 3$.
9. Name two angles that are complements of $\angle HOF$.
10. Why are $\angle COE$ and $\angle BOG$ congruent?
11. Why are $\angle AOH$ and $\angle COE$ congruent?
12. Why are $\angle AOG$ and $\angle EOD$ congruent?
13. Why are $\angle 3$ and $\angle AOC$ congruent?
14. Why are $\angle 3$ and $\angle AOD$ supplements?

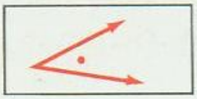
B.

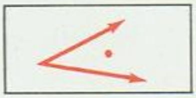
15. If $m\angle A = x$, then the complement of $\angle A$ has measure $\underline{?}$.
16. If $m\angle B = x$, then the supplement of $\angle B$ has measure $\underline{?}$.
17. Two angles are supplementary. The measure of one is four times the measure of the other. Find the measures of the two angles.
18. Two angles are supplementary. The measure of one is 20 less than three times the measure of the other. Find the measures of the two angles.

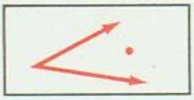
Activity

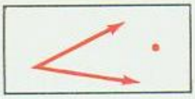
Make a set of 20 cards like the ones below. Clip them together, and flip through them to show that the locus of points equidistant from the two sides of an angle is the angle bisector.







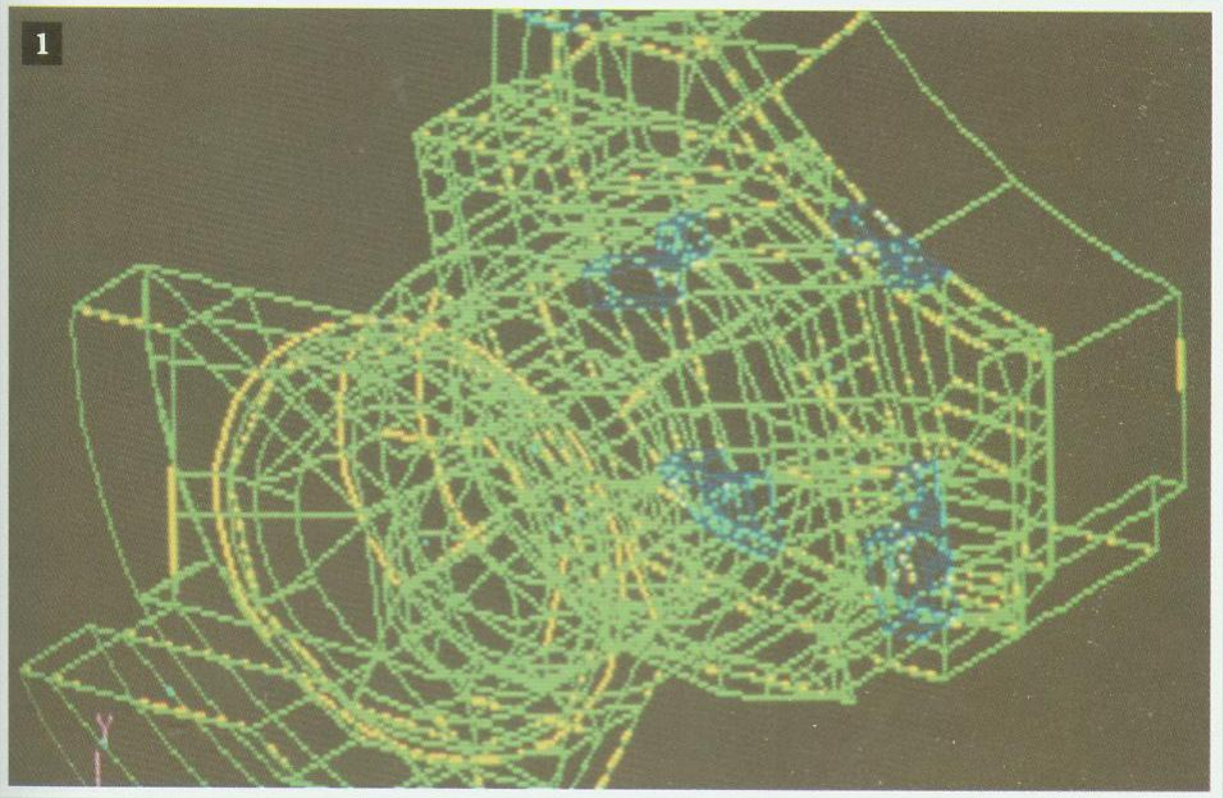




Computer Generated Geometry

Special programs make it increasingly easy to generate geometric diagrams on the computer screen. The geometry of machine parts and other technological devices can be analyzed with an accuracy never before possible. Geometry is revealed in places where it was unsuspected in science, business, and manufacturing. These photos show how simple circular shapes, parallel lines, intersecting lines, and planes work together to make visually exciting electronic artwork.

1 The geometry of this automobile part is apparent. Engineers can rotate the picture and view it from any angle.



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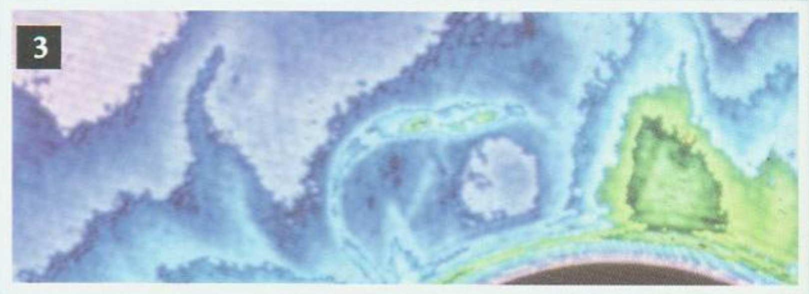
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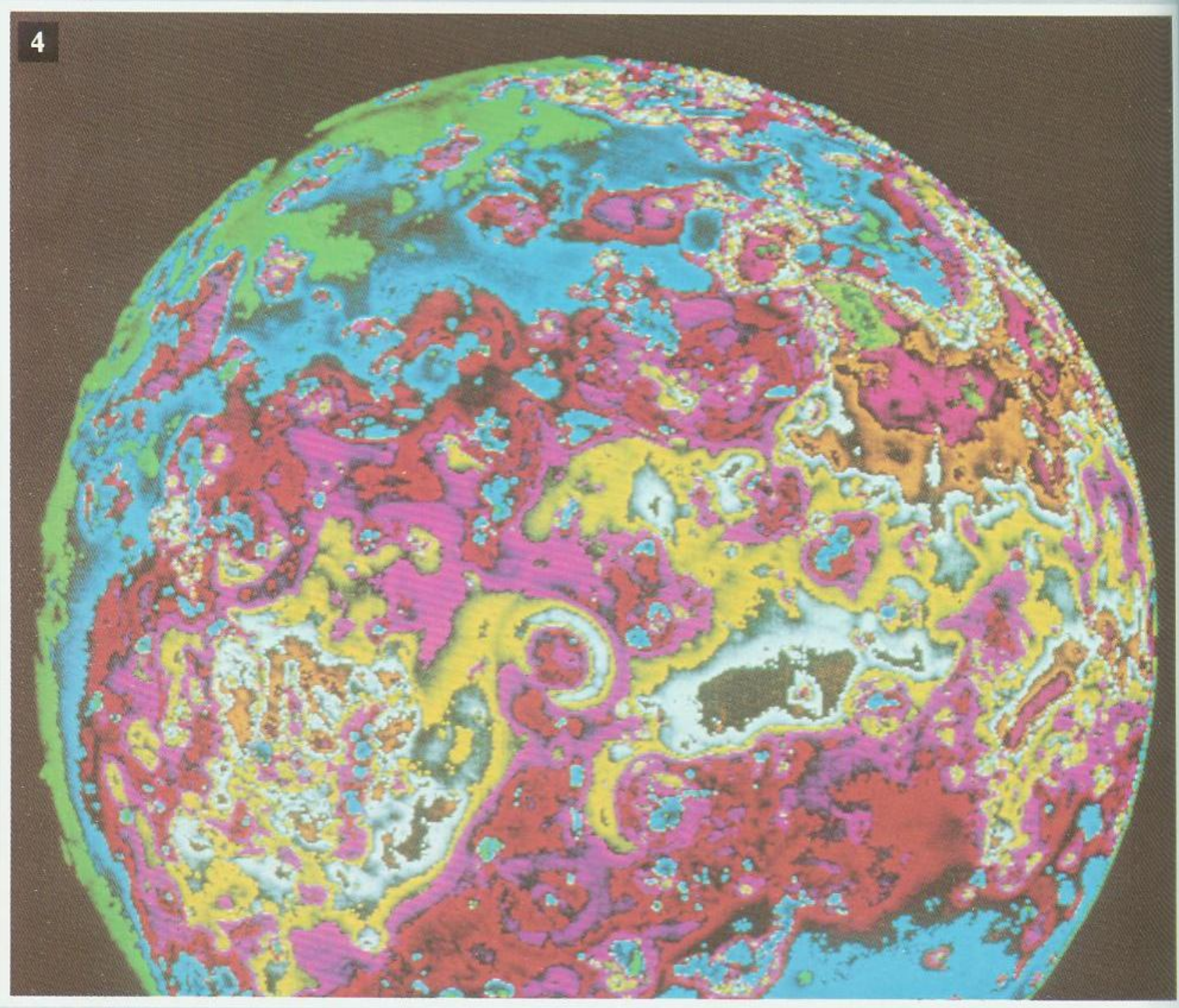

4

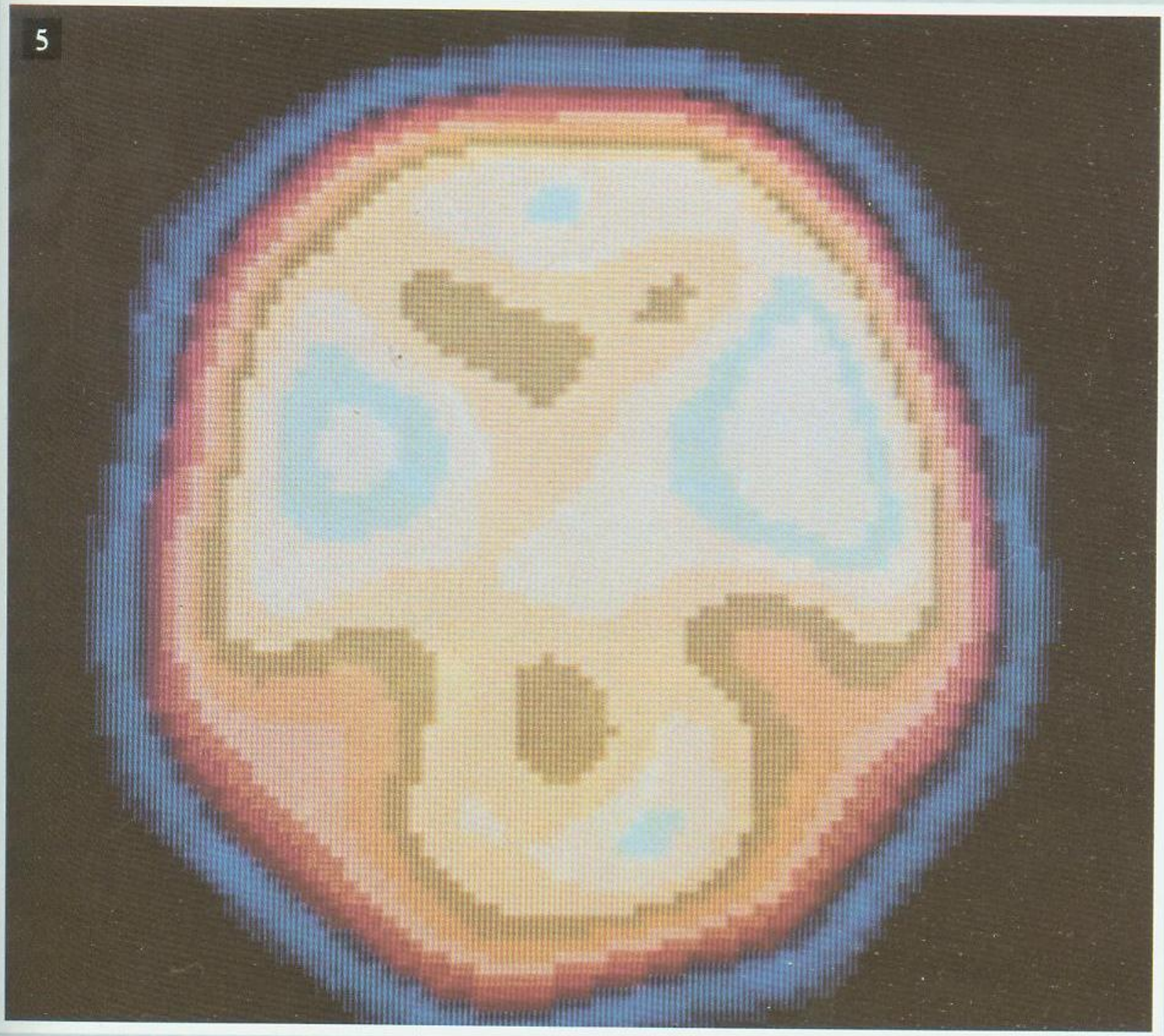
2 The curvature shown in this computer-produced drawing of Io, one of Jupiter's moons, can be used to calculate its diameter.

3 Actual colors from this explosion on the sun are translated by computers into this photograph to show differences in temperature.

4 This photo of one of Jupiter's moons was radioed to earth and processed by computer to show its geometry more clearly.

5 A computerized brain scanner provides a "map" permitting researchers to study brain activity. This photo shows a normal human brain.

5



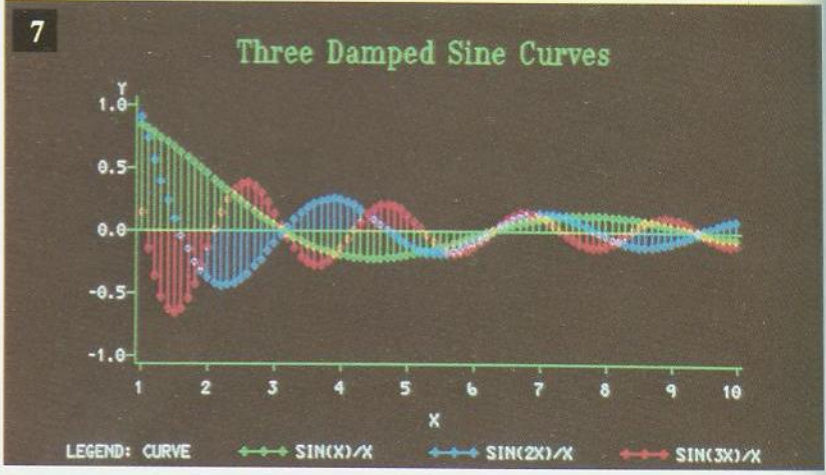
6 An example of abstract design shows the computer's capability.

7 The geometry of these curves is important in analyzing business activity.

8 The surface in this graph looks like a three-dimensional image of a cowboy hat.

9 The graph above is now shown in a two-dimensional countour plot.

7



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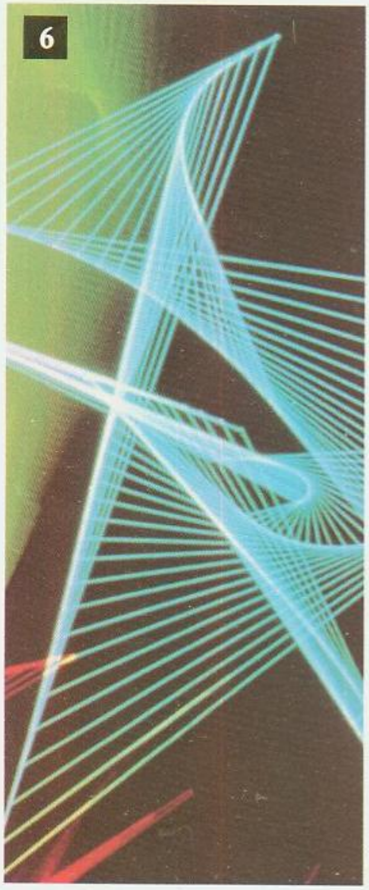

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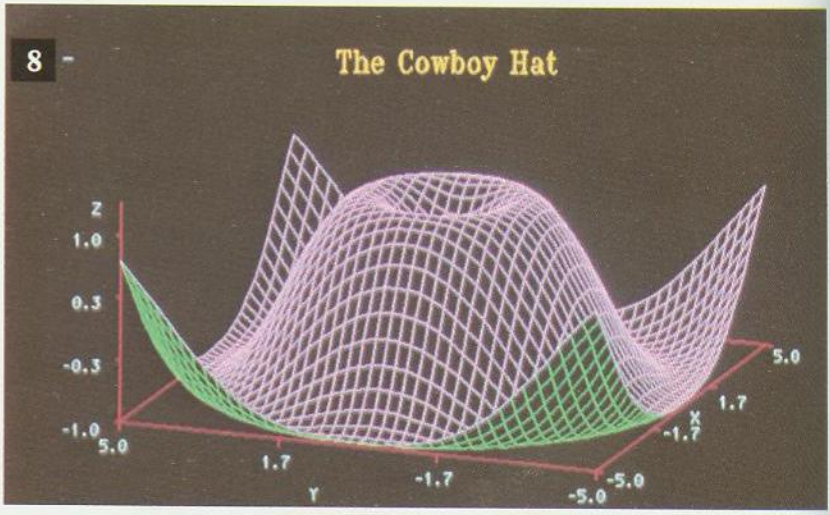

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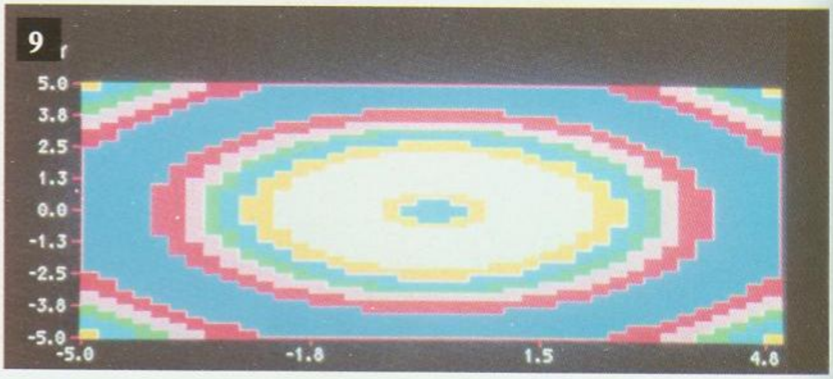


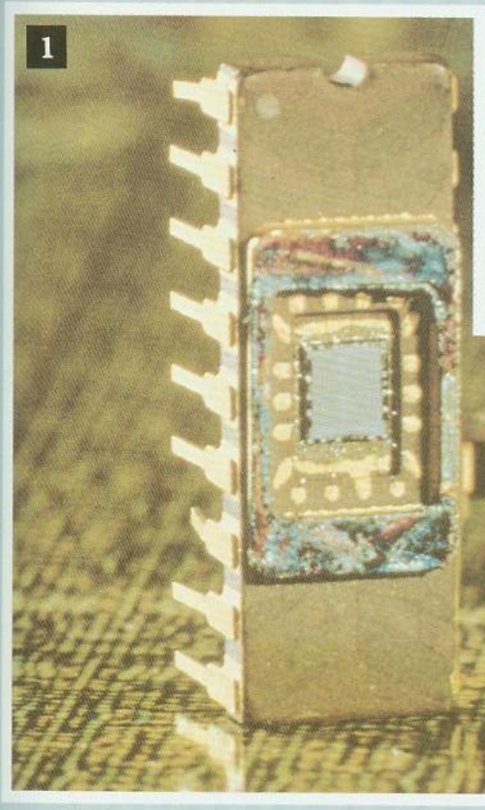
The Geometry of a Silicon Chip

A silicon chip, the heart of a computer, is a miracle of miniaturization. Electrical circuits are drawn by engineers and then photographically reduced 500 times smaller. Then the circuits are reproduced on a sheet of silicon for use inside the computer. In these photos the geometric concepts of congruence, similarity, parallel, and perpendicular are all strikingly apparent.

1 **The gray rectangular piece mounted on this carrier is a silicon chip. Mass-produced chips are identical in their finest detail.**

2 **The size of a chip and its carrier are dramatically illustrated.**

1



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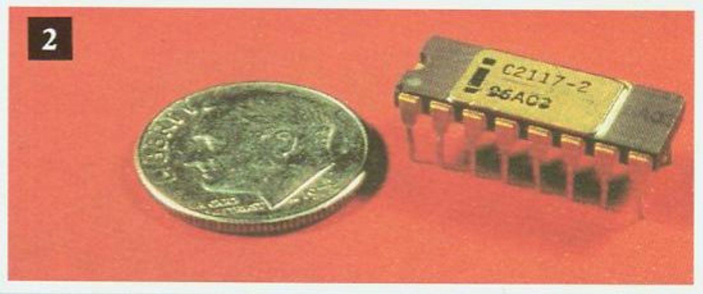

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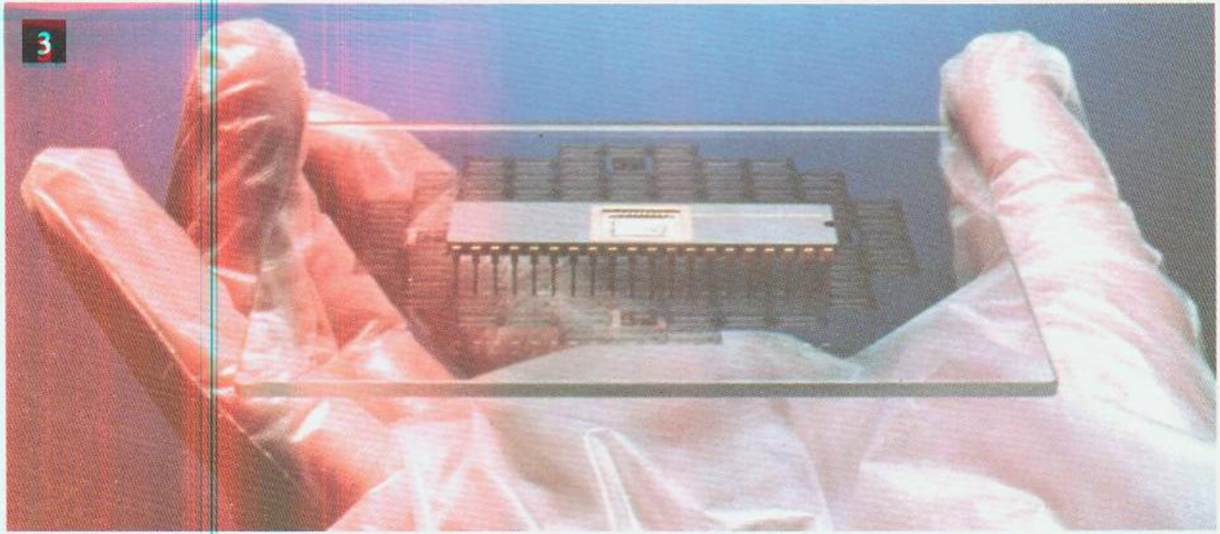

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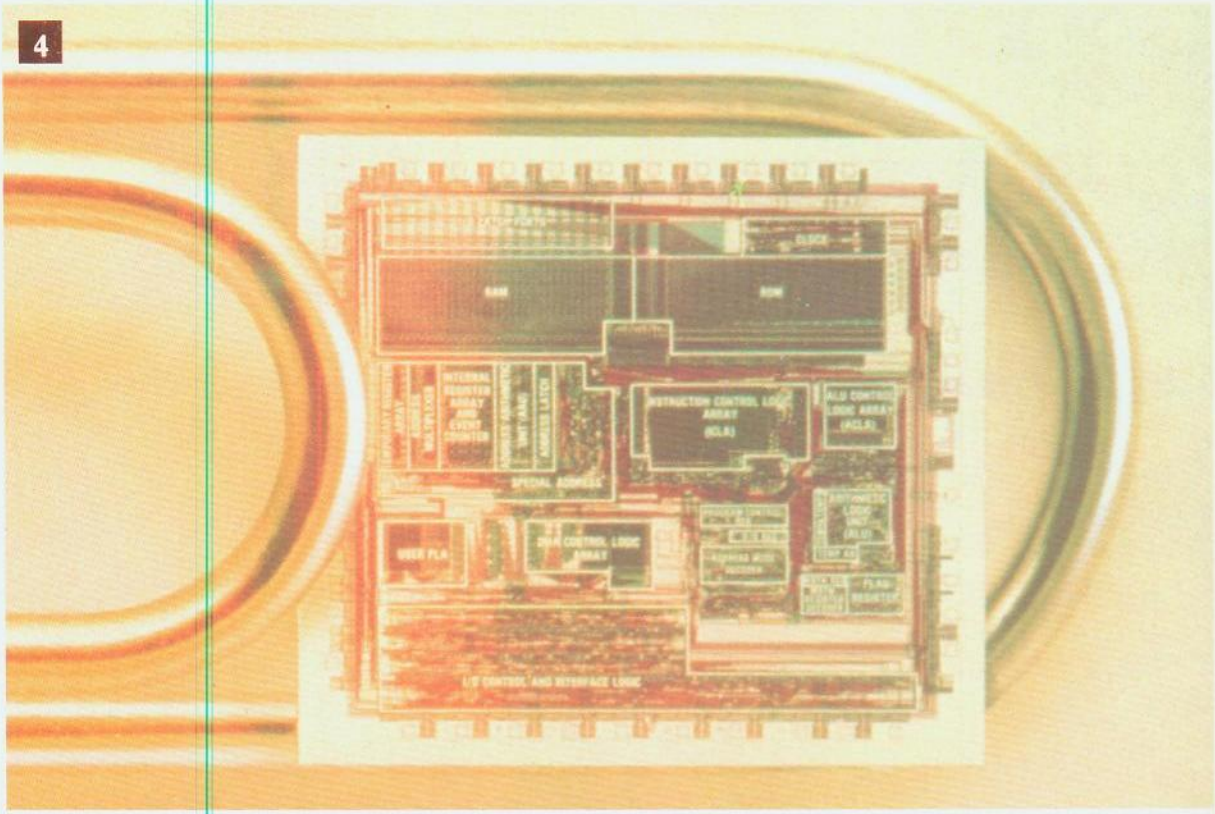
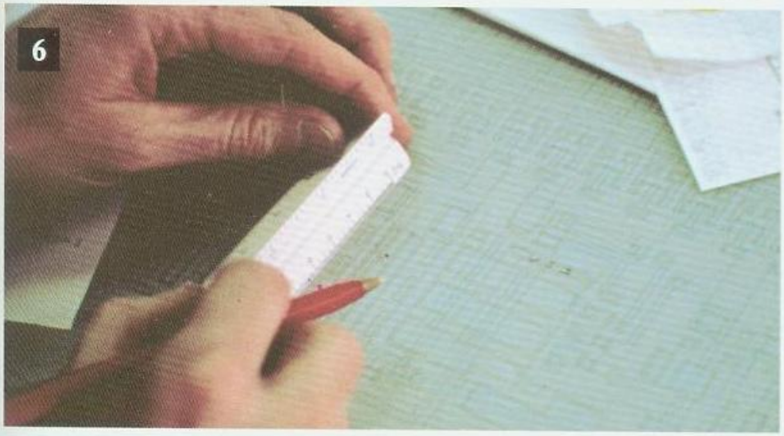

CLOCK
RAM
ROM
INTERNAL REGISTER ARRAY AND EVENT COUNTER
ADDRESS MULTIPLEXER
ADDRESS ARITHMETIC UNIT (AAU)
ADDRESS LATCH
INSTRUCTION CONTROL LOGIC ARRAY (ICLA)
ALU CONTROL LOGIC ARRAY (ACLA)
SPECIAL ADDRESS
ARITHMETIC LOGIC UNIT (ALU)
USER PLA
DMA CONTROL LOGIC ARRAY
FLAG REGISTER
I/O CONTROL AND INTERFACE LOGIC

5

6



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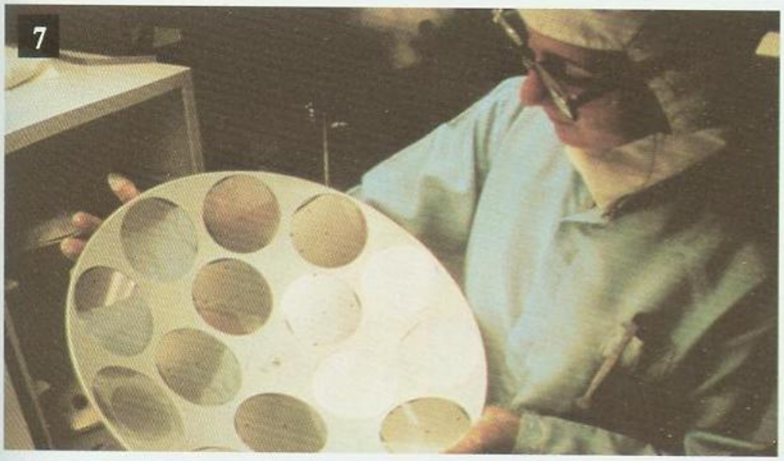


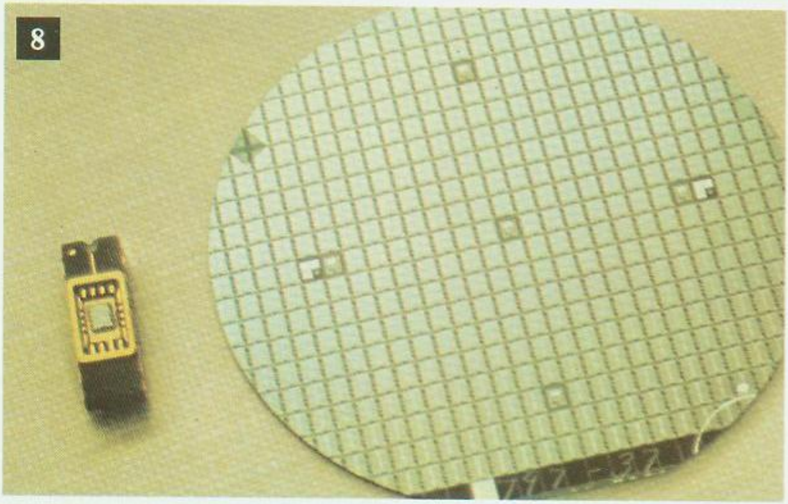
3 This chip and its holder are ready to be plugged into a printed circuit board along with other components to do useful work.

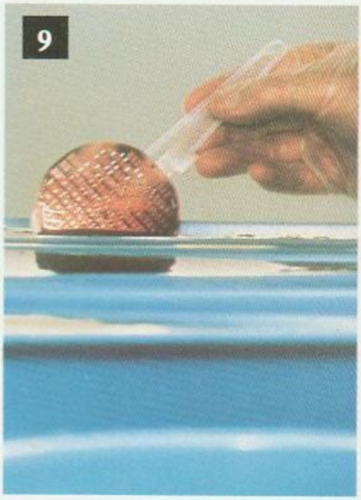
4 Space visualization skills are used in creating chip designs. Component parts are shaped so that they fit together with little wasted space.

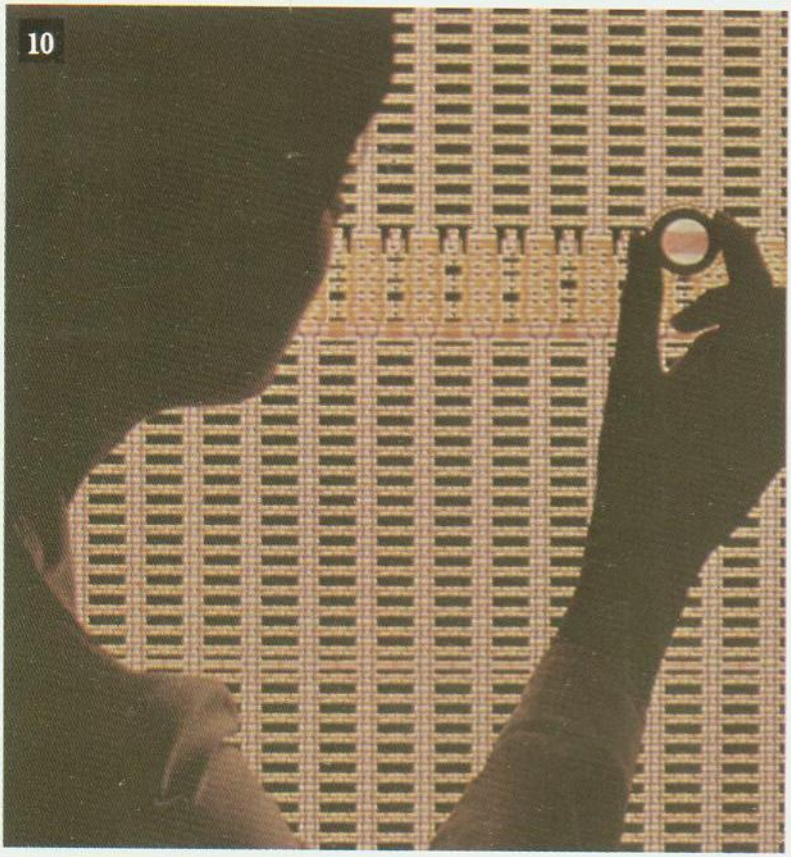
5 Difficult space visualization problems must be solved to design such an intricate network.

6 Original drawings as large as 8 feet by 8 feet are made. Notice how polygon shapes are joined with parallel and perpendicular electrical impulse paths.

7 The large diagram is photographically reduced and manufactured onto these disks like a sheet of postage stamps.







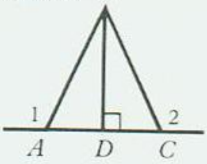
8 A disk of chips is compared to a single chip. Notice that the disk is a tesselation of chips.

9 Here, only one layer of circuitry is complete. As many as ten layers may be placed one on top of another to make one chip. The circuit designer has a very interesting geometric puzzle.

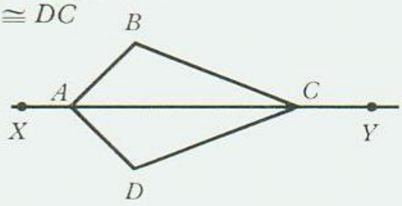
10 A technician checks circuits before they are reduced. Any flaw must be detected at this stage. Notice the role of symmetry and repeating patterns in circuit design.

Use Theorems 4-7 and 4-8 in exercises 19-22.

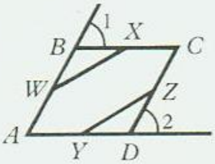
19. Given: $\overline{AD} \cong \overline{CD}$
$\overline{BD} \perp \overline{AC}$
$\angle 1 \cong \angle 2$
Prove: $\triangle ABD \cong \triangle CBD$



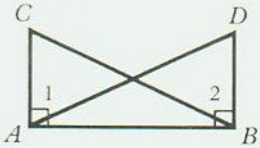
20. Given: $\angle BAX \cong \angle DAX$
$\angle BCY \cong \angle DCY$
Prove: $\overline{BC} \cong \overline{DC}$



21. Given: $ABCD$ is a quadrilateral with all sides equal in length.
$\angle 1 \cong \angle 2$
W, X, Y, Z are midpoints of the sides.
Prove: $\triangle BWX \cong \triangle DZY$

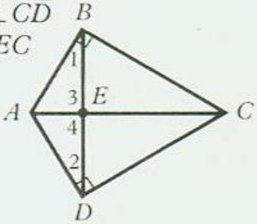


22. Given: $\overline{AC} \perp \overline{AB}$, $\overline{BD} \perp \overline{AB}$
$\angle 1 \cong \angle 2$
Prove: $\overline{AD} \cong \overline{BC}$

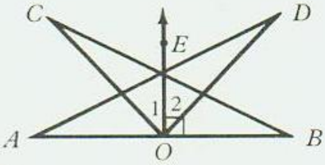


C.

23. Given: $\angle 1 \cong \angle 2$
$\angle 3 \cong \angle 4$
$\overline{BE} \cong \overline{DE}$
$\overline{AB} \perp \overline{BC}$, $\overline{AD} \perp \overline{CD}$
Prove: $\triangle BEC \cong \triangle DEC$



24. Given: $\overline{AB} \perp \overrightarrow{OE}$, O is the midpoint of $\overline{AB}$.
$\angle A \cong \angle B$
$\angle 1 \cong \angle 2$
Prove: $\triangle AOD \cong \triangle BOC$

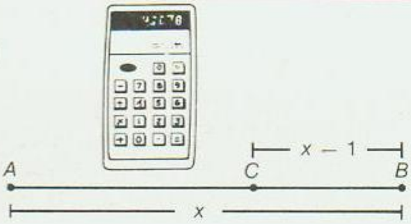


25. Prove the second part of Theorem 4-7.

26. Prove Theorem 4-8.

PROBLEM SOLVING

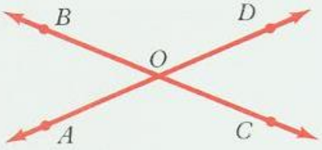
1. Use a calculator and a "guess and check" method to find two decimals x and y such that $x - y = 1$ and $x \cdot y = 1$.
2. If $\frac{AB}{AC} = \frac{AC}{CB}$ this line is divided into a special ratio called the *golden section*. Show that AB is the same as the number x you found in problem 1.



4–4 Proving Theorems—Using Vertical Angles

In this lesson we study pairs of angles formed by a pair of intersecting lines.

The windmill shown here provides many examples of angles formed by intersecting lines. In addition to many linear pairs of angles, there are many pairs of angles that are across from each other. These are called vertical angles. The theorems in this lesson are about the relationships between angles formed by intersecting lines.



$\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ intersect at O.

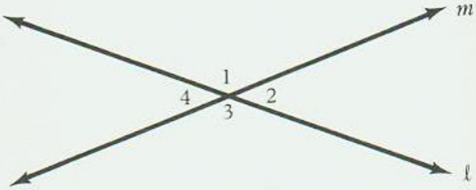
$\angle AOB$ and $\angle COD$ are vertical angles.

Definition 4–4

Vertical angles are two angles which are formed by two intersecting lines but which are not a linear pair of angles.

Can you convince someone that these five statements are true about the angles formed by the intersecting lines ℓ and m?

1. $\angle 1$ and $\angle 2$ are supplementary.
2. $\angle 3$ and $\angle 2$ are supplementary.
3. $\angle 1$ and $\angle 3$ are congruent.
4. $\angle 3$ and $\angle 4$ are supplementary.
5. $\angle 2$ and $\angle 4$ are congruent.



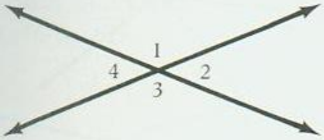
The situations above lead to the following theorem.

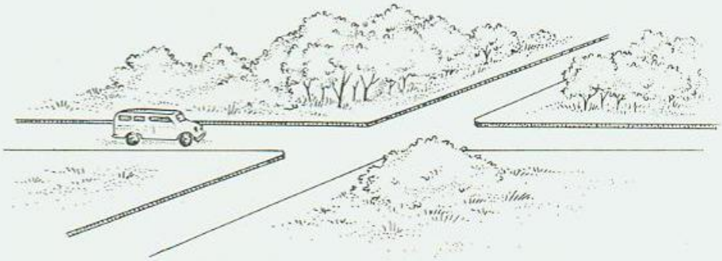
Theorem 4–9 **Vertical Angles Theorem.** If two lines intersect, the vertical angles are congruent.

PROOF

Step 1 This step is already completed by the statement of the theorem.

Step 2





Step 3 Given: Two pairs of vertical angles,
$\angle 1$ and $\angle 3$
$\angle 2$ and $\angle 4$

Step 4 Prove: $\angle 1 \cong \angle 3$, and
$\angle 2 \cong \angle 4$

Step 5 I shall prove that $\angle 1 \cong \angle 3$ by using the fact that $\angle 1$ and $\angle 2$ form a linear pair, and that $\angle 2$ and $\angle 3$ form a linear pair. Then I shall use the Linear Pair Postulate and the Congruent Supplements Theorem.

The proof that $\angle 2 \cong \angle 4$ will be identical so I shall not repeat the proof a second time.

Step 6

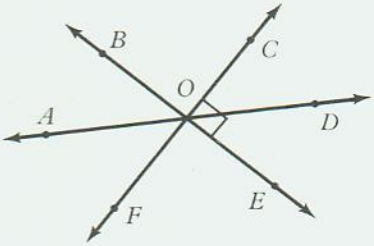
Statements	Reasons
1. $\angle 1$ and $\angle 3$ are vertical angles	1. Given
2. $\angle 1$ and $\angle 2$ form a linear pair	2. Definition of linear pair
3. $\angle 1$ is supplementary to $\angle 2$	3. Linear Pair Postulate
4. $\angle 3$ and $\angle 2$ form a linear pair	4. Why?
5. $\angle 3$ is supplementary to $\angle 2$	5. Why?
6. $\angle 1 \cong \angle 3$	6. Congruent Supplements Theorem

The following theorem also relates to linear pairs of angles.

Theorem 4–10 If one angle of a linear pair is a right angle, then the other is also a right angle.

EXERCISES

A.



(Exs. 1-4)

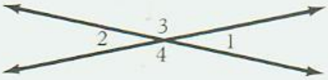
1. Name two pairs of right angles that are vertical angles.

2. Name a pair of acute angles that are vertical angles.

3. Name a pair of obtuse angles that are vertical angles.

4. Name six pairs of congruent angles.

If $m\angle 2 = 35$, give the measures in exercises 5-8.



(Exs. 5-8)

5. $m\angle 3 = \underline{?}$

6. $m\angle 1 = \underline{?}$

7. $m\angle 4 = \underline{?}$

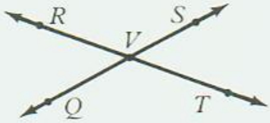
8. $m\angle 1 + m\angle 4 = \underline{?}$

Use the information given in exercises 9-11 to find the angle measures.

9. $m\angle RVQ = 4x$
   $m\angle SVT = 2x + 20$
   $m\angle RVQ = \underline{?}$

10. $m\angle QVT = 5x$
    $m\angle RVS = 8x - 45$
    $m\angle RVS = \underline{?}$

11. $m\angle RVQ = 2x + 30$
    $m\angle SVT = 3x + 20$
    $m\angle SVT = \underline{?}$

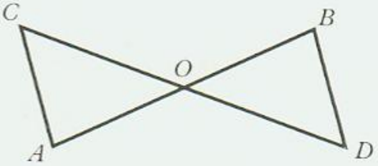


(Exs. 9-11)

B.

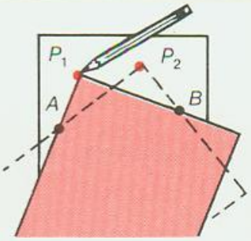
Write two-column proofs for these exercises. Often Theorem 4-9 can be used.

12. **Given:** $\overline{AB}$ and $\overline{CD}$ intersecting at O
    $\overline{AO} \cong \overline{OB}$
    $\angle A \cong \angle B$
    Prove: $\triangle AOC \cong \triangle BOD$



Activity

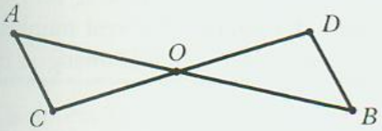
Mark two points A and B on your paper and lay down another sheet of paper as shown so that one side of the sheet is on A and the other side on B. Mark red point P_1. Do this again with the paper in another position and mark point P_2. Continue this until you have marked 20 different points. What is the locus of points P_1, P_2, P_3, P_4, P_5, P_6, etc.?



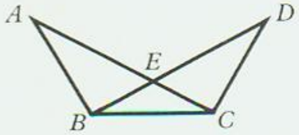
13. **Given:** $\overline{AB}$ and $\overline{CD}$ intersecting at O
$\overline{AB}$ bisects $\overline{CD}$.
$\angle C \cong \angle D$
Prove: $\triangle AOC \cong \triangle BOD$

14. **Given:** $\overline{AB}$ and $\overline{CD}$ intersect at their midpoints.
Prove: $\overline{AC} \cong \overline{BD}$

15. **Given:** $\overline{AE} \cong \overline{DE}$
$\overline{BE} \cong \overline{CE}$
Prove: $\overline{AB} \cong \overline{CD}$



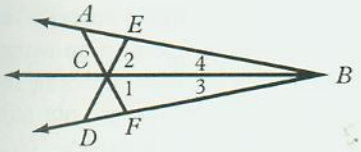
(Exs. 13, 14)

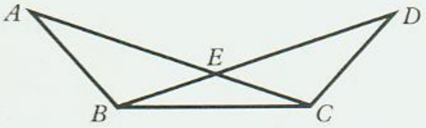


C.

16. **Given:** $\overrightarrow{BC}$ bisects $\angle ABD$.
$\angle 1 \cong \angle 2$
Prove: $\triangle ABC \cong \triangle DBC$

17. **Given:** $\overline{AE} \cong \overline{DE}$
$\overline{BE} \cong \overline{CE}$
Prove: $\triangle ABC \cong \triangle DCB$



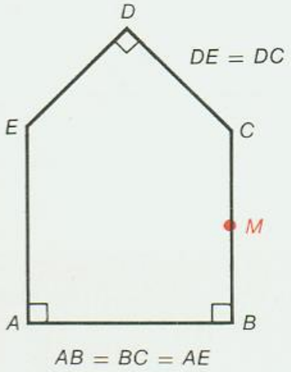


18. **Prove:** If two angles in a linear pair are congruent, then the angles are right angles.

19. **Prove:** If an angle is congruent to its supplement, then the angle is a right angle.

PROBLEM SOLVING

Suppose you have a board shaped as shown. You want to saw it into three pieces that can be rearranged to form a square. How can this be done, using two cuts? (*Hint:* Use the midpoint, M, of $\overline{BC}$.)



4–5 Proving Theorems—Using Exterior Angles

REVIEW: Some Inequality Properties of Real Numbers

Definition of Greater Than: $a > b$ means that $a = b + c$ and c is a positive number.

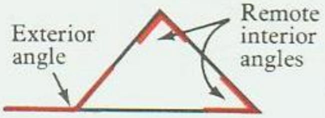
Transitive Property: If $a > b$ and $b > c$, then $a > c$.

Addition Property: If $a > b$, then $a + c > b + c$.

Multiplication Property: If $a > b$ and $c > 0$, $ac > bc$. If $a > b$ and $c < 0$, $ac < bc$.

Trichotomy Property: For real numbers a and b, one and only one of the following is true: $a = b$, $a > b$, or $a < b$.

The number properties listed above will be used to prove the theorem in this lesson. First, we need two definitions.

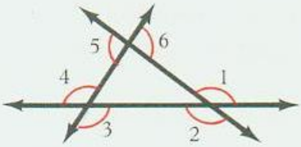


Definition 4-5

An **exterior angle** of a triangle is an angle that forms a linear pair with one of the angles of the triangle.

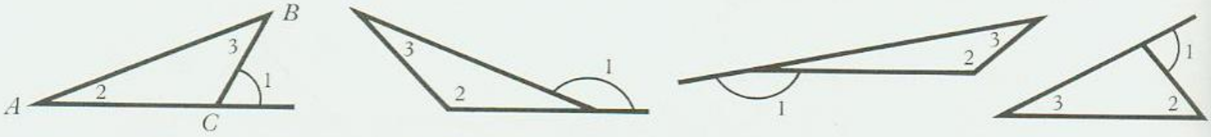
Definition 4-6

The **remote interior angles** with respect to an exterior angle are the two angles of the triangle that are not adjacent to the exterior angle.



Note that each triangle has six exterior angles, as shown in this figure.

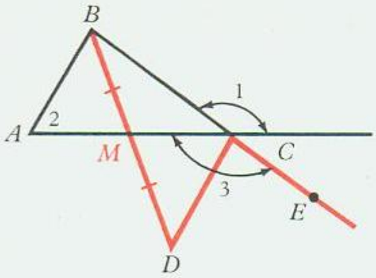
Trace exterior angle 1 and compare it with the remote interior angles in each triangle below.



The comparisons lead to the following theorem:

Theorem 4–11 **The Exterior Angle Theorem.** The measure of an exterior angle of a triangle is greater than the measure of either remote interior angle.

PROOF



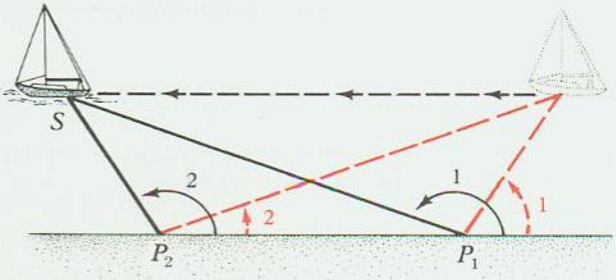
Given: $\triangle ABC$ with exterior angle, $\angle 1$
Prove: $m\angle 1 > m\angle 2$

Plan: At this stage in your study of geometry, you would not be expected to plan a proof like this.

Statements	Reasons
1. $\angle 1$ is an exterior angle of $\triangle ABC$.	**1.** Given
2. Let M be the midpoint of $\overline{AC}$.	**2.** Choice of M
3. On $\overleftrightarrow{BM}$ choose point D such that $\overline{BM} \cong \overline{MD}$.	**3.** Point-Line Postulate, choice of D
4. $\overline{AM} \cong \overline{MC}$	**4.** Definition of midpoint
5. $\angle BMA \cong \angle DMC$	**5.** Why?
6. $\triangle AMB \cong \triangle CMD$	**6.** Why?
7. $\angle MCD \cong \angle 2$	**7.** CPCTC
8. $m\angle MCD = m\angle 2$	**8.** Definition of angle congruence
9. $m\angle MCD + m\angle DCE = m\angle 3$	**9.** Definition of between for rays
10. $m\angle 2 + m\angle DEC = m\angle 3$	**10.** Substitution
11. $m\angle 3 > m\angle 2$	**11.** Definition of greater than
12. $m\angle 1 = m\angle 3$	**12.** Why?
13. $m\angle 1 > m\angle 2$	**13.** Substitution

APPLICATION

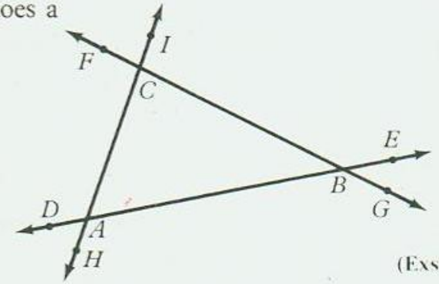
Two observers, at points P_1 and P_2, watch a ship sail by. The angles their lines of sight make with the shore line ($\angle 1$, $\angle 2$) are constantly changing. $m\angle 1$ appears to be greater than $m\angle 2$ in both ship positions shown. Will $m\angle 1$ always be greater than $m\angle 2$ as the ship sails much further down the coast line?



Answer. Because of the Exterior Angle Theorem, we know that $m\angle 1 > m\angle 2$, no matter what the location of the ship. Thus, the angle the observers' lines of sight make with the shore is always greater for the observer at point P_1.

EXERCISES

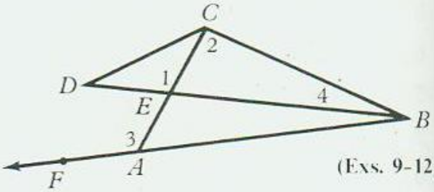
A.



(Exs. 1-8)

1. Name the exterior angles. How many exterior angles does a triangle have in all?
2. Name the remote interior angles of $\angle DAC$.
3. Name the remote interior angles of $\angle HAB$.
4. $\angle ABC$ is a remote interior angle for which exterior angles?
5. Is $\angle DAH$ an exterior angle? Why?
6. What is the relationship of $\angle ICB$ to $\angle CBA$?
7. What is the relationship of $\angle ACB$ to $\angle FCI$?
8. What is the relationship of $\angle ABG$ to $\angle ABC$?

B.

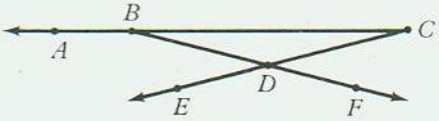


(Exs. 9-12)

Explain why each statement is true in exercises 9–12.

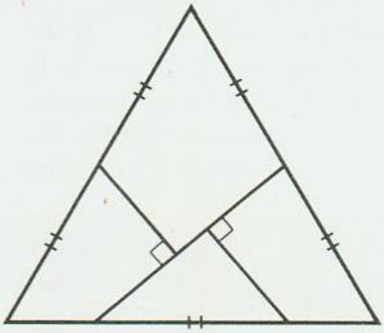
9. $m\angle 1 > m\angle 2$
10. $m\angle 3 > m\angle 1$
11. $m\angle 3 > m\angle 4$
12. $m\angle 3 > m\angle 2$

13. Use the figure to prove that $m\angle ABD > m\angle EDF$.



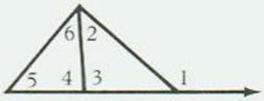
Activity

Trace and cut an equilateral triangle as shown and arrange the pieces to form a square.

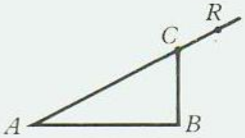


14. Use this figure to prove that $m\angle 1 > m\angle 6$.

15. **Given:** $\triangle ABC$ is a triangle with right angle $\angle B$.
Prove: $\angle RCB$ is an obtuse angle.



(Ex. 14)



(Ex. 15)

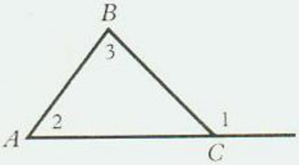
C.

Use the Exterior Angle Theorem to prove the theorems in exercises 16–17.

16. A triangle with one right angle has two acute angles.

17. If a triangle has one obtuse angle, then the other two angles are acute.

18. On page 155, a proof is given that $m\angle 1 > m\angle 2$. Use a similar method and prove that $m\angle 1 > m\angle 3$. (*Hint:* Use the midpoint of $\overline{BC}$.)



(Ex. 18)

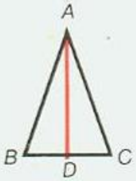
19. Prove that a triangle cannot have both a right angle and an obtuse angle.

PROBLEM SOLVING

Auxiliary lines are sometimes introduced to help solve a problem. But we have to be sure that the lines that are introduced exist. Consider the following proof.

"Theorem": Every triangle is isosceles.

Given: $\triangle ABC$
Prove: $\triangle ABC$ is isosceles.



Statements	Reasons
1. Draw in the line through A and D, the midpoint of $\overline{BC}$, that is perpendicular to $\overline{BC}$.	**1.** Construction of $\overline{AD}$
2. $\overline{AD} \cong \overline{AD}$	**2.** Reflexive property of congruence
3. $\overline{BD} \cong \overline{DC}$	**3.** Def. of midpoint
4. $\angle ADB \cong \angle ADC$	**4.** Def. of perpendicular
5. $\triangle ADB \cong \triangle ADC$	**5.** SAS Postulate
6. $\overline{AB} \cong \overline{AC}$	**6.** Corresponding parts
7. $\triangle ABC$ is isosceles	**7.** Def. of isosceles

Explain what is wrong with the proof. Draw a counterexample to show that statement 1 describes a line that does not always exist.

4–6 Using Indirect Proof

The method of Direct Proof used so far involves starting with given conditions and deducing that a conclusion is true. Indirect Proof is illustrated by what might happen in this cartoon—if B.C. cooperates! Suppoose B.C. eats the berry but does not die. An indirect proof that the berry is not poison could be the following.

> Suppose the berry is poison.
> If B.C. eats it, he will die.
> B.C. ate it, but didn't die.
> Therefore, the berry is not poison!

This method of proof involves assuming the negation of what you want to prove. Using reasoning, we show that the assumption leads to a contradiction. Thus, the statement to be proved must be true.

HERE, EAT THIS.
WHAT FOR?
I WANT TO FIND OUT IF IT'S POISON.
WHAT HAPPENS IF IT IS?
3-24
WE NAME A CEMETERY AFTER YOU.
hart

B.C. by permission of Johnny Hart and Field Enterprises, Inc.

We summarize the steps for indirect proof below:

Steps for Proving a Theorem Using Indirect Proof

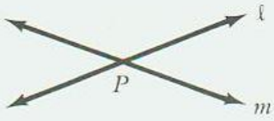
Step 1 Write the "GIVEN" and "PROVE" from the hypothesis and conclusion of the if-then statement.

Step 2 Assume the negation of the "PROVE" statement. (INDIRECT PROOF ASSUMPTION)

Step 3 Write the steps in the proof to show that the assumption leads to the contradiction of a known fact (theorem, definition, given information, etc.).

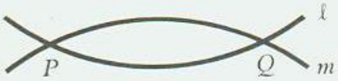
Step 4 Conclude that the assumption is false and that the "PROVE" statement is true.

We illustrate Indirect Proof by proving two theorems.



Theorem: If two lines intersect, the intersection is one point.

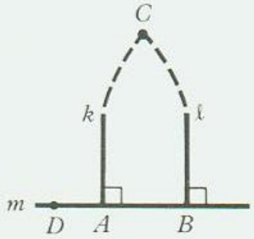
Given: Lines ℓ and m intersect at a point P.
Prove: P is the only point of intersection.
} **Step 1**



Analysis: Step 2 of an indirect proof is to assume the negation of the "Prove" statement. The negation of "P is the only point of intersection" is "P is *not* the only point of intersection." This means that there is a second point of intersection. Then we show that having two such points leads to a contradiction of a postulate.

	Statements	Reasons
Step 2	1. Suppose P is not the only point of intersection of lines ℓ and m.	1. Indirect proof assumption
Step 3	2. Call the second point of intersection Q.	2. Restatement of 1
	3. ℓ contains both P and Q. m contains both P and Q.	3. Statements 1 and 2
	4. P and Q are contained in two lines (contradiction of Point-Line Postulate).	4. Restatement of 3
Step 4	5. Therefore, P is the only point of intersection of ℓ and m.	5. Logic of indirect proof

Theorem: If two lines are perpendicular to the same line, the two lines are parallel.



Given: $k \perp m$, $\ell \perp m$
Prove: $k \parallel \ell$
} **Step 1**

	Statements	Reasons
Step 2	1. Assume $k \nparallel \ell$ (k is not parallel to ℓ)	1. Indirect proof assumption
	2. k intersects ℓ at point C.	2. Restatement of 1
Step 3	3. $\triangle ABC$ is formed.	3. Definition of triangle
	4. $m\angle DAC > m\angle ABC$	4. Exterior Angle Theorem
	5. $k \perp m$, $\ell \perp m$	5. Given
	6. $\angle DAC$ and $\angle ABC$ are right angles.	6. Definition of perpendicular lines
	7. $m\angle DAC = m\angle ABC$ (contradiction of $m\angle DAC > m\angle ABC$)	7. All right angles have equal measures.
Step 4	8. Therefore, $k \parallel \ell$	8. Logic of indirect proof

EXERCISES

A.

For each "Prove" statement in exercises 1-10, write the indirect proof assumption you would use to begin the proof, together with the second statement that interprets the indirect proof assumption.

Example: **Prove:** $\ell \parallel m$
Indirect proof assumption: $\ell \nparallel m$
Second statement: Then ℓ intersects m.

1. **Prove:** $\ell \perp m$
2. **Prove:** $\angle A$ is supplementary to $\angle B$.
3. **Prove:** The adjacent sides are not parallel.
4. **Prove:** $\angle A$ is not a right angle.
5. **Prove:** $\overline{AB} \cong \overline{CD}$
6. **Prove:** $\angle A$ and $\angle B$ are not vertical angles.
7. **Prove:** $\angle A$ is an acute angle.
8. **Prove:** There is one and only one line through P and parallel to m.
9. **Prove:** $\triangle ABC$ is an isosceles triangle.
10. **Prove:** $\triangle ABC$ is not an equilateral triangle.

Which pairs of statements in exercises 11-16 would enable you to arrive at a contradiction in an indirect proof?

Examples: $\left.\begin{array}{l}\overline{AB} \text{ is longer than } \overline{CD}, \text{ and} \\ \overline{CD} \text{ is longer than } \overline{AB}.\end{array}\right\}$ form a contradiction

$\left.\begin{array}{l}\triangle ABC \text{ is equilateral and} \\ \triangle ABC \text{ is isosceles.}\end{array}\right\}$ do not form a contradiction

11. Lines p and q are parallel and lines p and q do not intersect.
12. $\angle A \cong \angle B$ and $m\angle A > m\angle B$.
13. $\ell \perp m$ and $\ell \not\perp m$. (note: $\not\perp$ means "is not perpendicular to")
14. $\angle A$ and $\angle B$ form a linear pair. $m\angle A < 90$ and $m\angle B < 90$.
15. $\angle A$ is a right angle.
    $\angle A$ is an obtuse angle.
16. $\angle A$ and $\angle B$ are congruent.
    $\angle A$ and $\angle B$ are supplementary.

For exercises 17 and 18 select the statement that is contradictory to the given statement.

17. $\angle A$ and $\angle B$ are supplementary.

a. $m\angle A + m\angle B = 180$
b. $\angle A$ and $\angle B$ form a linear pair.
c. $\angle A$ and $\angle B$ are both acute angles.
d. $\angle A$ and $\angle B$ are vertical angles.

18. Lines p and q are not parallel.

a. Lines p and q have no points in common.
b. Lines p and q intersect.
c. Lines p and q are the same line.
d. Lines p and q lie in a plane and have no points in common.

Write a statement that contradicts the given statement in each of exercises 19–23.

19. $\angle A$ and $\angle B$ are complementary.
20. Lines p and q intersect.
21. $\triangle ABC \cong \triangle XYZ$
22. $ABCDE$ is a regular pentagon.
23. $m\angle A = 117$

For exercises 24–27, write the assumption that would be used in an indirect proof of the given theorem.

24. If two lines do not intersect, then they are not perpendicular.

25. If $AB \neq CD$, then $AB + EF \neq CD + EF$.

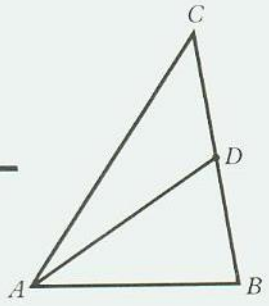
26. If a figure is a triangle, then the figure cannot contain two right angles.

27. If two angles form a linear pair, then they are supplementary.

B.

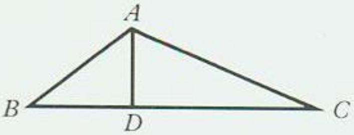
In exercises 28–30 an indirect proof is started. Give the missing reasons and continue the proof to reach a contradiction.

28. Given: In $\triangle ABC$, $\overline{AC} \not\cong \overline{AB}$
D is the midpoint of $\overline{BC}$.
Prove: $\overline{AD}$ cannot be perpendicular to $\overline{BC}$.



Statements	Reasons
1. Suppose $\overline{AD} \perp \overline{BC}$.	**1.** Indirect proof assumption
2. $\angle ADC \cong \angle ADB$	**2.** ?
3. D is midpoint of $\overline{BC}$.	**3.** Given
4. $\overline{CD} \cong \overline{BD}$	**4.** ?
5. $\overline{AD} \cong \overline{AD}$	**5.** ?
6. $\triangle ADC \cong \triangle ADB$	**6.** SAS Postulate
7. $\overline{AC} \cong \overline{AB}$	**7.** CPCTC
8. ?	**8.** ?
9. ?	**9.** ?

29. Given: $\overline{AD} \perp \overline{BC}$
$\overline{AB} \not\cong \overline{AC}$
Prove: $\overline{BD} \not\cong \overline{DC}$



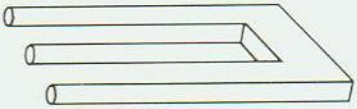
Statements	Reasons
1. Suppose $\overline{BD} \cong \overline{DC}$.	1. ?
2. $\overline{AD} \perp \overline{BC}$	2. ?
3. $\angle BDA \cong \angle ADC$	3. Definition of perpendicular
4. $\overline{AD} \cong \overline{AD}$	4. ?
5. $\triangle ADB \cong \triangle ADC$	5. ?
6. ?	6. ?
7. ?	7. ?
8. ?	8. ?

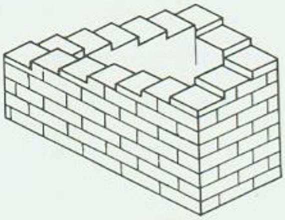
Activity

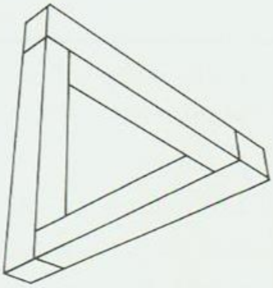
Either the object pictured can exist in the real world or it cannot. Suppose it can . . . ?

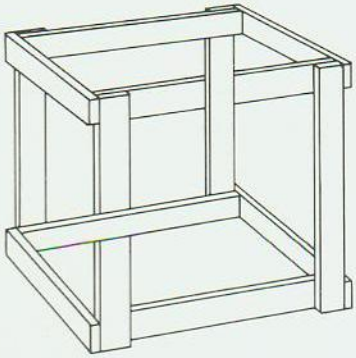
Continue the above argument for each figure and see if you can reach a contradiction of real-world facts.

Look for other pictures of objects such as these which are contradictory.

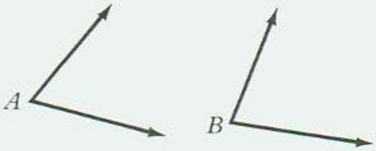








30. Given: $\angle A$ and $\angle B$ are acute.
Prove: $\angle A$ and $\angle B$ are not supplementary.



Statements	Reasons
1. Suppose $\angle A$ is supplementary to $\angle B$.	1. ?
2. $m\angle A + m\angle B = 180$	2. Definition of supplementary
3. $m\angle A < 90$, $m\angle B < 90$	3. Given, definition of acute angle
4. ?	4. Addition Property of Inequality
5. ?	5. ?

C.

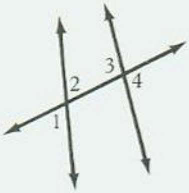
31. How might a lawyer use indirect proof? Give an example.

Write a complete indirect proof for each theorem in exercises 32-35.

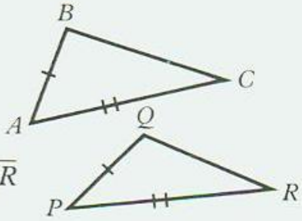
32. If $AB \neq CD$, then $AB + EF \neq CD + EF$.

33. If $\angle A$ is not congruent to $\angle B$, then $\angle A$ and $\angle B$ are not vertical angles.

34. **Given:**
$\angle 1$ and $\angle 4$ are not supplementary.
Prove:
$\angle 2$ and $\angle 3$ are not supplementary.



35. **Given:**
$\overline{AB} \cong \overline{PQ}$
$\overline{AC} \cong \overline{PR}$
$\angle A \ncong \angle P$
Prove: $\overline{BC} \ncong \overline{QR}$



PROBLEM SOLVING

When an advertisement about a bargain radio for sale said "It's a steal," the store owner didn't realize how true it was. She was convinced that either Ann, Bob, Cal, or Dee had stolen the radio. Each person, in turn, made a statement, but only one of the four statements was true.

Ann said "I didn't take it."
Bob said "Ann is lying."
Cal said "Bob is lying."
Dee said "Bob took it."

Who told the truth? Who took the radio?

Important Ideas—Chapter 4

Terms

Complementary angles (p. 144)
Supplementary angles (p. 144)
Linear pair of angles (p. 144)
Vertical angles (p. 150)
Exterior angle of a triangle (p. 154)
Remote interior angle (p. 154)

Postulate

Linear Pair Postulate. If two angles form a linear pair, the angles are supplementary.

Theorems

4-1 The reflexive, symmetric, and transitive properties hold for angle and segment congruence.

4-2 In an isosceles triangle the segment from the vertex angle to the midpoint of the opposite side forms a pair of congruent triangles.

4-3 **Addition of Equal Angles.** If $m\angle APB = m\angle DQE$, $m\angle BPC = m\angle EQF$, $\overrightarrow{PB}$ is between $\overrightarrow{PA}$ and $\overrightarrow{PC}$, and $\overrightarrow{QE}$ is between $\overrightarrow{QD}$ and $\overrightarrow{QF}$, then $m\angle APC = m\angle DQF$.

4-4 **Subtraction of Equal Segments.** If $AC = DF$, $BC = EF$, B is between A and C, and E is between D and F, then $AB = DE$.

4-5 **Addition of Equal Segments.** If $AB = DE$, $BC = EF$, B is between A and C, and E is between D and F, then $AC = DF$.

4-6 **Subtraction of Equal Angles.** If $m\angle APC = m\angle DQF$, $m\angle BPC = m\angle EQF$, $\overrightarrow{PB}$ is between $\overrightarrow{PA}$ and $\overrightarrow{PC}$, and $\overrightarrow{QE}$ is between $\overrightarrow{QD}$ and $\overrightarrow{QF}$, then $m\angle APB = m\angle DQE$.

4-7 **Congruent Complements Theorem.** Two angles that are complementary to the same angle (or congruent angles) are congruent.

4-8 **Congruent Supplements Theorem.** Two angles that are supplementary to the same angle (or to congruent angles) are congruent.

4-9 **Vertical Angles Theorem.** If two lines intersect, the vertical angles are congruent.

4-10 If one angle of a linear pair is a right angle, then the other is also a right angle.

4-11 **Exterior Angle Theorem.** The measure of an exterior angle is greater than the measure of either remote interior angle.

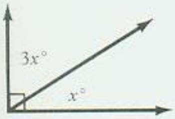
Chapter 4—Review

1. Indicate whether the following statements are true or false.
   - **a.** If two angles are supplementary to congruent angles, then they are congruent.
   - **b.** The transitive property for congruence of segments states that $\overline{AB} \cong \overline{AB}$.
   - **c.** The sum of the measures of two complementary angles is 90°.

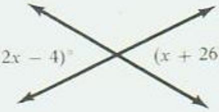
In exercises 2 and 3 a theorem is given in if-then form. Draw a picture and state the "Given" and "Prove" using the picture and its labeling.

2. If two angles of a triangle are congruent, then the triangle is isosceles.
3. If two lines are parallel to a third line then they are parallel.
4. Solve for x and find the measure of the angles.

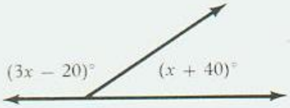
a.



b.

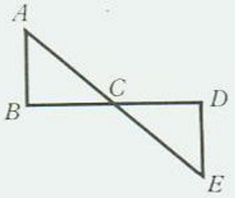


c.



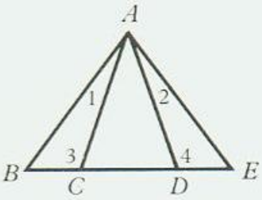
5. What is the measure of an angle that is congruent to its complement?

6. **Given:** $\overline{AB} \perp \overline{BD}$, $\overline{DE} \perp \overline{BD}$, $BC = CD$
   Prove: $\triangle ABC \cong \triangle EDC$



7. **Given:** $m\angle 1 = m\angle 2$
   Prove: $m\angle BAD = m\angle CAE$

8. **Given:** $\overline{AB} \cong \overline{AE}$, $\angle E \cong \angle B$, and $\angle 1 \cong \angle 2$
   Prove: $\angle ACD \cong \angle ADC$



(Exs. 7, 8)

9. For the following "Prove" statements, write the indirect proof assumption you would use to begin the proof.
   - **a.** $\overline{AB} \parallel \overline{CD}$
   - **b.** $\angle A$ and $\angle B$ are supplementary.

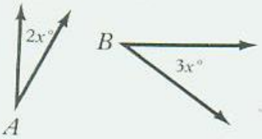
Chapter 4—Test

1. Indicate whether the following statements are true or false.
   - **a.** The measure of an exterior angle of a triangle is greater than the measure of any angle of the triangle.
   - **b.** If two angles are both supplementary and congruent, then each angle is a right angle.
   - **c.** If two angles are supplementary then they form a linear pair.

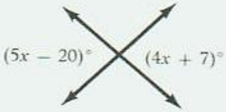
In exercises 2 and 3 a theorem is given in if-then form. Draw a picture and state the "Given" and "Prove" using the picture and its labeling.

2. If a triangle is equilateral, then the measure of each angle is 60°.
3. If a line segment joins the midpoints of two sides of a triangle, then the segment is parallel to the third side.
4. Solve for x and find the measures of the angles.

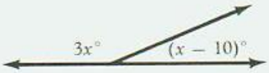
a. $\angle A$ and $\angle B$ are complementary



b.

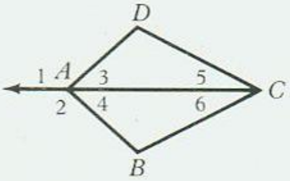


c.

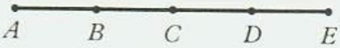


5. What is the measure of an angle whose measure is half the measure of its supplement?

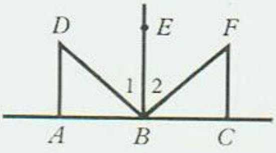
6. **Given:** $m\angle 1 = m\angle 2$
   $m\angle 5 = m\angle 6$
   Prove: $AD = AB$



7. **Given:** $AB = CD$, $BD = CE$
   Prove: $AC = CE$



8. **Given:** $\angle 1 \cong \angle 2$, $\overline{DA} \perp \overline{AC}$, $\overline{EB} \perp \overline{AC}$ and $\overline{FC} \perp \overline{AC}$, B is midpoint of $\overline{AC}$.
   Prove: $\overline{AD} \cong \overline{CF}$

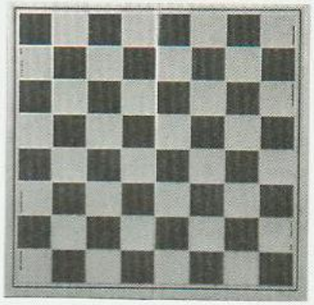


9. For the following "Prove" statements, write the indirect proof assumption you would use to begin the proof.
   - **a.** $\overline{AB} \cong \overline{CD}$
   - **b.** $\angle DEF$ is not a right angle

Problem Solving Techniques

Make a Table-I

A useful technique in solving problems is to make a table and see if a pattern can be observed. Study the example below.



How many squares can you find?

Example

How many squares are on an 8 × 8 checkerboard?

Consider the table on the right. The number of 1 × 1, 2 × 2, 3 × 3, and 4 × 4 squares are given. Can you observe a pattern and solve the problem?

Size of square	Number of squares
1 × 1	64
2 × 2	49
3 × 3	36
4 × 4	25
5 × 5	?
6 × 6	?
7 × 7	?
8 × 8	?

Solution. The number of 1 × 1 squares is 8^2 squares. The number of 2 × 2 squares is 7^2. The pattern appears to be decreasing perfect squares. So we fill in the table with 16, 9, 4 and 1. To get the total we add and find that it is 204 squares.

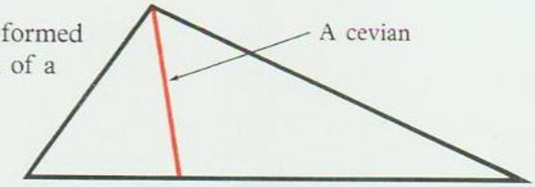
PROBLEMS

For each problem, make a table, observe a pattern, and solve the problem.

1. How many cubes are in an 8 × 8 × 8 cube? (*Hint:* Make a table similar to the one above. One column lists the size of the cubes: 1 × 1 × 1, 2 × 2 × 2, 3 × 3 × 3, etc. The second column gives the number of each sized cubes.)
2. How many diagonals does a ten-sided polygon have? (*Hint:* Make a table like the one on the right.)

Sides	Diagonals
3	0
4	2
5	?
6	?
⋮	⋮

3. A cevian is a line segment which joins a vertex of a triangle and a point on the opposite side. How many triangles are formed if 8 cevians are drawn from one vertex of a triangle?



(Note that one cevian gives 3 triangles.)